Motivic Serre group, algebraic Sato-Tate group and Sato-Tate conjecture

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Abstract. We make explicit Serre's generalization of the Sato-Tate conjecture for motives, by expressing the construction in terms of fiber functors from the motivic category of absolute Hodge cycles into a suitable category of Hodge structures of odd weight. This extends the case of abelian varietes, which we treated in a previous paper. That description was used by Fité–Kedlaya–Rotger–Sutherland to classify Sato-Tate groups of abelian surfaces; the present description is used by Fité–Kedlaya–Sutherland to make a similar classification for certain motives of weight 3. We also give conditions under which verification of the Sato-Tate conjecture reduces to the identity connected component of the corresponding Sato-Tate group.

1. Introduction

In [**Se2**], Serre gave a general approach, in terms of the motivic category for numerical equivalence, towards the question of equidistribution of Frobenius elements in families of l-adic representations; this approach puts such questions as the Chebotarev density theorem and the Sato-Tate conjecture in a common framework. Serre revisited this topic in [**Se3**], making the description somewhat more explicit. The purpose of this paper is to follow in this direction, expressing Serre's construction in terms of fiber functors from the motivic category of absolute Hodge cycles into a suitable category of Hodge structures of odd weight. This extends our previous paper [**BK**], in which we carried out this program for abelian varieties; this was motivated by the immediate application to the classification of Sato-Tate groups of abelian surfaces in [**FKRS**]. Similarly, the results of this paper are used in [**FKS**] to carry out a similar classification for a special class of motives of weight 3, and are expected to find further use in similar classifications for other classes

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of motives of odd weight. (Some modifications are needed to handle cases of even weight, such as K3 surfaces.) The organization of the paper is as follows.

In chapter 2, we briefly recall some facts about Hodge structures and Mumford-Tate groups in a fashion suitable for our exposition.

In chapter 3, we extend the notion of twisted decomposable Lefschetz group, introduced in [**BK**], to Hodge structures with some extra endomorphism structure. The twisted decomposable Lefschetz group (Definition 3.4) is the disjoint sum of Galois twists (Definition 3.3) of the Lefschetz group.

In chapter 4, we work with Hodge structures associated with families of l-adic representations and prove basic results concerning relations between group schemes $G_{l,K,1}^{\mathrm{alg}}$ and $G_{l,K}^{\mathrm{alg}}$.

In chapter 5, we state the algebraic Sato-Tate conjecture for families of l-adic representations associated with Hodge structures. We also restate the Sato-Tate conjecture in this case and prove some basic properties of the algebraic Sato-Tate group and the Sato-Tate group. In particular, under the algebraic Sato-Tate conjecture, we establish the isomorphism (Proposition 5.7) between the groups of connected components of the algebraic Sato-Tate and Sato-Tate groups. We also introduce Galois twists inside $G_{l,K,1}^{\mathrm{alg}}$ (see Definition 5.12) and we explain the relation of these twists to Galois twists of the corresponding Lefschetz group.

In chapter 6, under some mild assumptions on the base field K, we compute connected components of $G_{l,K,1}^{\mathrm{alg}}$ (Theorem 6.11). Then, under the algebraic Sato-Tate conjecture, we make a corresponding computation of connected components of AST_K and ST_K (Theorem 6.12). As a consequence, we prove that the Sato-Tate conjecture holds with respect to ST_K if and only if it holds with respect to the connected component of ST_K (Theorem 6.12).

In chapter 7, we show how to compute Mumford-Tate and Hodge groups for powers of Hodge structures and similarly how to compute $G_{l,K,1}^{\mathrm{alg}}$ and $G_{l,K}^{\mathrm{alg}}$ for powers of l-adic representations. We also observe that in some cases, the Mumford-Tate conjecture implies the algebraic Sato-Tate conjecture.

In chapter 8, we continue the discussion from chapter 7 of the relationship between the algebraic Sato-Tate conjecture and the Mumford-Tate conjecture. We establish conditions for the algebraic Sato-Tate conjecture to hold with the algebraic Sato-Tate group equal to the corresponding twisted decomposable Lefschetz group.

Chapters 9–11 give the application of chapters 2–8 to the case where the polarized Hodge structures and associated l-adic representations come from motives in the motivic category of absolute Hodge cycles introduced by Deligne [**D1**], [**DM**]. All the assumptions on Hodge structures and associated l-adic representations we made in chapters 2–5 are satisfied in this case.

At the beginning of chapter 9, we recall some results concerning the category $\mathcal{M}_K$ of motives for absolute Hodge cycles. Next, for a motive M of $\mathcal{M}_K$ we introduce the Artin motive $h^0(D)$ corresponding to $D := \mathrm{End}_{\mathcal{M}_{\overline{K}}}(\overline{M})$ and compute the motivic Galois group $G_{\mathcal{M}_K^0(D)}$ of the smallest Tannakian subcategory $\mathcal{M}_K^0(D)$ of $\mathcal{M}_K$ generated by $h^0(D)$. Also, let $\mathcal{M}_K(M)$ denote the smallest Tannakian subcategory of $\mathcal{M}_K$ generated by M. From this point on in the paper, we work only

with *homogeneous motives*, i.e., motives which occur as factors of motives of the form $h^r(X)(m)$ for some smooth projective variety X over K and some and $m \in \mathbb{Z}$. For M a homogeneous motive, we consider the motivic Galois group $G_{\mathcal{M}_K(M)}$ and the motivic Serre group $G_{\mathcal{M}_K(M),1}$ (Definition 9.5). We also define Galois twists (Definition 9.10) inside the motivic Serre group and explain their relation to Galois twists of the corresponding Lefschetz group. The precise computation of $G_{\mathcal{M}^0_K(D)}$ allows us to write down the motivic Serre group as a disjoint union of these twists (see (9.29)).

At the beginning of chapter 10, we find a sufficient condition (Theorem 10.2) for the natural map $\pi_0(G_{\mathcal{M}_K(M),1}) \to \pi_0(G_{\mathcal{M}_K(M)})$ to be an isomorphism. Theorem 10.2 is the motivic analogue of Theorem 4.8. We then introduce the motivic Mumford-Tate conjecture and motivic Sato-Tate conjecture.

We start chapter 11 by recalling the relationship between the motivic Mumford-Tate group with the corresponding Mumford-Tate group and the relation of the motivic Serre group with the Hodge group. Under Serre's conjecture that $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$, i.e., that the Mumford-Tate group is equal to the connected component of the motivic Mumford-Tate group, we define (Definition 11.7) the algebraic Sato-Tate group. We collect the main properties of the algebraic Sato-Tate group in Theorem 11.8. At the end of this chapter, under the assumption that $\mathrm{H}(V,\psi) = C_D(\mathrm{Iso}_{(V,\psi)})$, we show that the algebraic Sato-Tate group is the corresponding twisted decomposable Lefschetz group (Corollary 11.10). In addition, under the Mumford-Tate conjecture, we prove the algebraic Sato-Tate conjecture in this case (Corollary 11.11). We finish by proving, under an assumption on homotheties in the associated l-adic representations and under the algebraic Sato-Tate Conjecture for the base field, that the Sato-Tate conjecture holds with respect to ST_K if and only if it holds with respect to the connected component of ST_K (Theorem 11.14). Theorem 11.14 may serve of use in proving cases of the Sato-Tate conjecture, by making it possible to avoid computations involving connected components of the Sato-Tate group.

In conclusion, recall that for Absolute Hodge Cycles (AHC) motives (Definition 11.3), Serre's conjecture $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$ holds (Remark 11.4). Hence the algebraic Sato-Tate group is defined (Definition 11.7) unconditionally for AHC motives. All motives associated with abelian varieties are AHC motives ([**D1**, Theorem 2.11]). Moreover, if $\mathcal{M}_K^{\mathrm{av}}$ denotes the Tannakian subcategory of $\mathcal{M}_K$ generated by abelian varieties and Artin motives, then every motive in $\mathcal{M}_K^{\mathrm{av}}$ is an AHC motive ([**DM**, Theorem 6.25]). So the algebraic Sato-Tate group is defined unconditionally for motives in $\mathcal{M}_K^{\mathrm{av}}$ (cf. [**BK**]). It is shown in [**DM**, Proposition 6.26] that the motives associated with curves, unirational varieties of dimension ≤ 3, Fermat hypersurfaces, and K3 surfaces belong to $\mathcal{M}_K^{\mathrm{av}}$. In general, the Hodge conjecture implies that every Hodge cycle on a motive is an algebraic cycle, and Deligne showed that every algebraic cycle is an absolute Hodge cycle ([**D1**, Example 2.1]). Hence the Hodge conjecture implies that every motive is an AHC motive.

2. Hodge structures and Mumford-Tate group

Let (V,ψ) be a rational, polarized, pure Hodge structure of weight n. Hence V is a vector space over $\mathbb{Q}$ and ψ is a bilinear nondegenerate $(-1)^n$ symmetric form $\psi : V \times V \to \mathbb{Q}(-n)$ such that $V_{\mathbb{C}}$ has a pure Hodge structure of weight n. Let

PHS($\mathbb{Q}$) denotes the category of rational, polarized, pure Hodge structures. The category PHS($\mathbb{Q}$) is abelian and semisimple [**D2**, Lemme 4.2.3, p. 44] (cf. [**PS**, Cor. 2.12, p. 40]). Let $D_h := D(V,\psi) := \mathrm{End}_{\mathrm{PHS}(\mathbb{Q})}(V,\psi)$. In particular D_h is a finite-dimensional semisimple algebra over $\mathbb{Q}$. If (V,ψ) is a simple polarized Hodge structure, then D_h is a division algebra. By the definition of a Hodge structure,

$$V_{\mathbb{C}} := V \otimes_{\mathbb{Q}} \mathbb{C} = \bigoplus_{n=p+q} V^{p,q} \tag{2.1}$$

where $\overline{V^{p,q}} = V^{q,p}$ and the $V^{p,q}$ are equivariant with respect to the action of the endomorphism algebra $D_h \otimes_{\mathbb{Q}} \mathbb{C}$. Put $\psi_{\mathbb{C}} := \psi \otimes_{\mathbb{Q}} \mathbb{C}$. Recall that $\mathbb{Q}(-n)$ is a pure Hodge structure of weight $2n$ such that $\mathbb{C}(-n) = \mathbb{C}(-n)^{n,n}$. The polarization ψ can be seen as the morphism $\psi : V \otimes_{\mathbb{Q}} V \to \mathbb{Q}(-n)$ of pure Hodge structures of weight $2n$ so that the $\mathbb{C}$-bilinear form $\psi_{\mathbb{C}} : V_{\mathbb{C}} \times V_{\mathbb{C}} \to \mathbb{C}(-n)$ has the property that $\psi_{\mathbb{C}}(V^{p,q} \times V^{p',q'}) = 0$ if $p+p' \neq n$ or $q+q' \neq n$.

Remark 2.1. More generally, (T,φ) is an integral, polarized, pure Hodge structure of weight n if T is a free abelian group and $\varphi : T \times T \to \mathbb{Z}(-n)$ is a nondegenerate $\mathbb{Z}$-bilinear map such that (V,ψ) is a rational, polarized pure Hodge structure of weight n, where $V := T \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\psi := \varphi \otimes \mathbb{Q}$. Let PHS($\mathbb{Z}$) denote the category of integral, polarized, pure Hodge structures.

Remark 2.2. A recent, simple approach to real Hodge and mixed Hodge structures can be found in [**BM1**], [**BM2**].

Remark 2.3. The vector space V defines a commutative group scheme, also denoted V by abuse of notation, whose points with values in a unital commutative $\mathbb{Q}$-algebra R are:

$$V(R) := V \otimes_{\mathbb{Q}} R.$$

If $d := \dim_{\mathbb{Q}} V$, then any choice of basis of the vector space V gives an isomorphism $V \cong \mathbb{A}^d$ of group schemes over $\mathbb{Q}$. We will equip V with the tautological action of the group scheme GL_V.

We will be particularly interested in those elements $g \in \mathrm{GL}_V$, for which there exists an element $\chi(g) \in \mathbb{G}_{m,\mathbb{Q}}$ such that $\psi(gv,gw) = \chi(g)\psi(v,w)$ for all $v,w \in V$. The following formulas determine group subschemes of GL_V of special interest.

$$\mathrm{GIso}_{(V,\psi)} := \{g \in \mathrm{GL}_V : \ \psi(gv,gw) = \chi(g)\psi(v,w) \ \ \forall\, v,w \in V\}, \tag{2.2}$$

$$\mathrm{Iso}_{(V,\psi)} := \{g \in \mathrm{GL}_V : \ \psi(gv,gw) = \psi(v,w) \ \ \forall\, v,w \in V\}. \tag{2.3}$$

There is also a map of group schemes

$$\chi : \mathrm{GIso}_{(V,\psi)} \to \mathbb{G}_{m,\mathbb{Q}}$$
$$g \mapsto \chi(g),$$

which is a character of $\mathrm{GIso}_{(V,\psi)}$ such that $\mathrm{Iso}_{(V,\psi)} = \mathrm{Ker}\,\chi$.

Remark 2.4. Observe that for every $\alpha \in \mathbb{G}_{m,\mathbb{Q}}$ by bilinearity of ψ we have:

$$\psi(\alpha\,\mathrm{Id}_V v,\, \alpha\,\mathrm{Id}_V w) = \psi(\alpha v,\, \alpha w) = \alpha^2\,\psi(v,\, w).$$

Hence

$$\alpha\,\mathrm{Id}_V \in \mathrm{GIso}_{(V,\psi)} \quad \text{and} \quad \chi(\alpha\,\mathrm{Id}_V) = \alpha^2. \tag{2.4}$$

We also observe that:

$$\mathrm{GIso}_{(V,\psi)} = \begin{cases} \mathrm{GO}_{(V,\psi)} & \text{if } \ n \ \text{even} \\ \mathrm{GSp}_{(V,\psi)} & \text{if } \ n \ \text{odd}; \end{cases} \tag{2.5}$$

$$\mathrm{Iso}_{(V,\psi)} = \begin{cases} \mathrm{O}_{(V,\psi)} & \text{if } \ n \ \text{even} \\ \mathrm{Sp}_{(V,\psi)} & \text{if } \ n \ \text{odd}. \end{cases} \tag{2.6}$$

Definition 2.5. For any pure Hodge structure (not necessarily polarized) define the cocharacter [**D1**, p. 42]

$$\mu_{\infty,V} \, : \, \mathbb{G}_m(\mathbb{C}) \to \mathrm{GL}(V_{\mathbb{C}}) \tag{2.7}$$

such that for any $z \in \mathbb{C}^{\times}$, the automorphism $\mu_{\infty,V}(z)$ acts as multiplication by z^{-p} on $V^{p,q}$ for each $p+q=n$.

Notice that the complex conjugate cocharacter is

$$\overline{\mu_{\infty,V}} \, : \, \mathbb{G}_m(\mathbb{C}) \to \mathrm{GL}(V_{\mathbb{C}}) \tag{2.8}$$

such that for any $z \in \mathbb{C}^{\times}$, the automorphism $\overline{\mu_{\infty,V}}(z)$ acts as multiplication by $\bar{z}^{-q}$ on $V^{p,q}$ for each $p+q=n$. Observe that for $v \in V^{p,q}$ and $w \in V^{n-p,n-q}$ we have:

$$\psi_{\mathbb{C}}(\mu_{\infty,V}(z)v, \, \mu_{\infty,V}(z)w) = \psi_{\mathbb{C}}(z^{-p}v, \, z^{p-n}w) = z^{-n}\psi_{\mathbb{C}}(v, \, w), \tag{2.9}$$

$$\psi_{\mathbb{C}}(\overline{\mu_{\infty,V}}(z)\, v, \, \overline{\mu_{\infty,V}}\,(z)w) = \psi_{\mathbb{C}}(\bar{z}^{-q}\, v, \, \bar{z}^{q-n}\, w) = \bar{z}^{-n}\psi_{\mathbb{C}}(v, \, w). \tag{2.10}$$

Hence

$$\mu_{\infty,V}(\mathbb{C}^{\times}) \, \subset \mathrm{GIso}_{(V,\,\psi)}(\mathbb{C}). \tag{2.11}$$

Since D_h commutes with $\mu_{\infty,V}(\mathbb{C}^{\times})$ on $V_{\mathbb{C}}$ elementwise, it is clear that:

$$\mu_{\infty,V}(\mathbb{C}^{\times}) \, \subset C_{D_h}\, \mathrm{GIso}_{(V,\,\psi)}(\mathbb{C}). \tag{2.12}$$

Let $\mathbb{S} := R_{\mathbb{C}/\mathbb{R}}\, \mathbb{G}_m$. The product $\mu_{\infty,V}\overline{\mu_{\infty,V}}$ restricted to each $V^{p,q} \oplus V^{q,p}$ gives the homomorphism of real algebraic groups:

$$h_{\infty,V} \, : \, \mathbb{S} \to \mathrm{GL}_{V_{\mathbb{R}}}\,. \tag{2.13}$$

It follows from (2.9), (2.10) that there is the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{Iso}_{(V_{\mathbb{R}},\psi_{\mathbb{R}})} \arrow[r] & \mathrm{GIso}_{(V_{\mathbb{R}},\psi_{\mathbb{R}})} \arrow[r] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{U}(1) \arrow[r] \arrow[u, "h_{\infty,V}"] & \mathbb{S} \arrow[r] \arrow[u, "h_{\infty,V}"] & \mathbb{G}_m \arrow[r] \arrow[u, "-n"] & 1
\end{tikzcd}$$

Definition 2.6. (Mumford-Tate and Hodge groups)

(1) The *Mumford-Tate group* of (V,ψ) is the smallest algebraic subgroup $\mathrm{MT}(V,\psi) \subset \mathrm{GIso}_{(V,\psi)}$ over $\mathbb{Q}$ such that $\mathrm{MT}(V,\psi)(\mathbb{C})$ contains $\mu_{\infty,V}(\mathbb{C})$.
(2) The *decomposable Hodge group* is $\mathrm{DH}(V,\psi) := \mathrm{MT}(V,\psi) \cap \mathrm{Iso}_{(V,\psi)}$.
(3) The *Hodge group* $\mathrm{H}(V,\psi) := \mathrm{DH}(V,\psi)^{\circ}$ is the connected component of the identity in $\mathrm{DH}(V,\,\psi)$.

We can equivalently define the Mumford-Tate and Hodge groups as follows.

(1) The *Mumford-Tate group* of (V,ψ) is the smallest algebraic subgroup $\mathrm{MT}(V,\psi) \subset \mathrm{GIso}_{(V,\psi)}$ over $\mathbb{Q}$ such that $\mathrm{MT}(V,\psi)(\mathbb{C})$ contains $h_{\infty,V}(\mathbb{S}(\mathbb{C}))$.
(2) The *Hodge group* of (V,ψ) is the smallest algebraic subgroup $\mathrm{H}(V,\psi) \subset \mathrm{Iso}_{(V,\psi)}$ over $\mathbb{Q}$ such that $\mathrm{H}(V,\psi)(\mathbb{C})$ contains $h_{\infty,V}(U(1)(\mathbb{C}))$.

Remark 2.7. Note that $\mathrm{MT}(V,\psi)$ is a reductive subgroup of $\mathrm{GIso}_{(V,\,\psi)}$ [**D1**, Prop. 3.6], [**PS**, Th. 2.19]. It follows by (2.12) that

$$\mathrm{MT}(V,\psi) \subset C_{D_h}(\mathrm{GIso}_{(V,\,\psi)}). \tag{2.14}$$

Moreover $\mathrm{H}(V,\psi) \subset \mathrm{Iso}_{(V,\,\psi)}$, hence

$$\mathrm{H}(V,\psi) \subset C_{D_h}(\mathrm{Iso}_{(V,\,\psi)}). \tag{2.15}$$

For additional background on Mumford-Tate groups, see the lecture notes of Moonen [**Mo1, Mo2**].

In our investigation of the algebraic Sato-Tate group for Hodge structures, we will need to investigate not only D_h but possibly also other subrings $D \subset \mathrm{End}_{\mathbb{Q}}(V)$ such that D acts on $V \otimes \mathbb{C}$ preserving the Hodge decomposition: $DV^{p,q} \subset V^{p,q}$ for all $p+q=n$. Such a D commutes with $\mu_{\infty,V}(\mathbb{C}^\times)$ on $V_{\mathbb{C}}$ elementwise, hence:

$$\mu_{\infty,V}(\mathbb{C}^\times) \subset C_D\, \mathrm{GIso}_{(V,\,\psi)}(\mathbb{C}). \tag{2.16}$$

Hence it follows by (2.16) that

$$\mathrm{MT}(V,\psi) \subset C_D(\mathrm{GIso}_{(V,\,\psi)}). \tag{2.17}$$

$$\mathrm{H}(V,\psi) \subset C_D(\mathrm{Iso}_{(V,\,\psi)}). \tag{2.18}$$

Definition 2.8. The algebraic group:

$$\mathrm{L}(V,\psi,D) := C_D^\circ(\mathrm{Iso}_{(V,\,\psi)}) \tag{2.19}$$

is called the *Lefschetz group* of (V,ψ) and the ring D.

Remark 2.9. By (2.18) and the connectedness of $\mathrm{H}(V,\psi)$, we have

$$\mathrm{H}(V,\psi) \subset \mathrm{L}(V,\psi,D). \tag{2.20}$$

In particular

$$\mathrm{H}(V,\psi) \subset \mathrm{L}(V,\psi,D_h). \tag{2.21}$$

3. Twisted Lefschetz groups

Let $D \subset \mathrm{End}_{\mathbb{Q}}(V)$ be a subring such that the action of D of $V \otimes \mathbb{C}$ preserves the Hodge decomposition, i.e. $DV^{p,q} \subset V^{p,q}$ for all p, q.

Fix a number field F and an algebraic closure $\overline{F}$. In this paper, K/F will denote any finite extension contained in $\overline{F}$. We assume that the ring D admits a continuous representation of the absolute Galois group G_F of F such that its restriction to G_K is denoted:

$$\rho_e\,:\,G_K \to \mathrm{Aut}_{\mathbb{Q}}(D). \tag{3.1}$$

Definition 3.1. The fixed field of the kernel of ρ_e will be denoted:

$$K_e \;:=\; \overline{K}^{\mathrm{Ker}\,\rho_e}.$$

Remark 3.2. The extension K_e/K is finite Galois and $G_{K_e} = \mathrm{Ker}\,\rho_e$. The field K_e depends on K; in particular, it is not invariant under base change along an arbitrary extension L/K. However, it is obvious that K_e will not change if we change base along an extension L/K such that $L \subset K_e$.

Definition 3.3. For $\tau \in \mathrm{Gal}(K_e/K)$, define:

$$\mathrm{DL}_K^{\tau}(V,\, \psi, D) := \{g \in \mathrm{Iso}_{(V,\, \psi)} :\, g\beta g^{-1} = \rho_e(\tau)(\beta) \;\; \forall\, \beta \in D\}. \tag{3.2}$$

Because D is a finite-dimensional $\mathbb{Q}$-vector space, $\mathrm{DL}_K^{\tau}(V,\psi,D)$ is a closed subscheme of $\mathrm{Iso}_{(V,\,\psi)}$ for each τ.

Definition 3.4. Define the *twisted decomposable algebraic Lefschetz group* for the triple $(V,\, \psi, D)$ to be the closed algebraic subgroup of $\mathrm{Iso}_{(V,\,\psi)}$ given by

$$\mathrm{DL}_K(V,\psi,D) \;:=\; \bigsqcup_{\tau\in \mathrm{Gal}(K_e/K)} \mathrm{DL}_K^{\tau}(V,\psi,D). \tag{3.3}$$

For any subextension L/K of $\overline{F}/K$, we have $\mathrm{DL}_L(V,\psi,D) \subseteq \mathrm{DL}_K(V,\psi,D)$ and $\mathrm{DL}_L^{\mathrm{id}}(V,\psi,D) = \mathrm{DL}_K^{\mathrm{id}}(V,\psi,D)$. Hence:

$$\begin{aligned} \mathrm{L}(V,\psi,D) = \mathrm{DL}_K^{\mathrm{id}}(V,\psi,D)^{\circ} = \mathrm{DL}_K(V,\psi,D)^{\circ} = \\ = \mathrm{DL}_L^{\mathrm{id}}(V,\psi,D)^{\circ} = \mathrm{DL}_L(V,\psi,D)^{\circ}. \end{aligned} \tag{3.4}$$

In particular,

$$\mathrm{DL}_{K_e}^{\mathrm{id}}(V,\psi,D) = \mathrm{DL}_{K_e}(V,\psi,D) = \mathrm{DL}_{\overline{F}}(V,\psi,D) = \mathrm{DL}_{\overline{F}}^{\mathrm{id}}(V,\psi,D), \tag{3.5}$$

$$\mathrm{L}(V,\psi,D) = \mathrm{DL}_{K_e}(V,\psi,D)^{\circ} = \mathrm{DL}_{\overline{F}}(V,\psi,D)^{\circ}. \tag{3.6}$$

Theorem 3.5. *The twisted decomposable Lefschetz groups have the following properties.*

1. $\mathrm{DL}_K^{\tau}(V^s,\psi^s,M_s(D)) \cong \mathrm{DL}_K^{\tau}(V,\psi,D)$ *for every* $\tau \in \mathrm{Gal}(K_e/K)$.
2. *Let* (V_i,ψ_i) *be polarized Hodge structures and let* D_i *be finite-dimensional* $\mathbb{Q}$*-algebras preserving the Hodge structures* V_i*. Let* D_i *admit a continuous* G_K*-action. Put* $(V,\psi) := \bigoplus_{i=1}^{t}(V_i,\psi_i)$ *and* $D := \prod_{i=1}^{t} D_i$*. Then* $\mathrm{DL}_K^{\tau}(V,\psi,D) \cong \prod_{i=1}^{t} \mathrm{DL}_K^{\tau}(V_i,\psi_i,D_i)$ *for every* $\tau \in \mathrm{Gal}(K_e/K)$.
3. *Let* (V_i,ψ_i) *be polarized Hodge structures and let* D_i *be finite-dimensional* $\mathbb{Q}$*-algebras preserving the Hodge structures* V_i*. Let* D_i *admit a continuous* G_K*-action. Put* $(V,\psi) := \bigoplus_{i=1}^{t}(V_i^{s_i},\psi_i^{s_i})$ *and* $D := \prod_{i=1}^{t} M_{s_i}(D_i)$*. Then* $\mathrm{DL}_K^{\tau}(V,\psi,D) \cong \prod_{i=1}^{t} \mathrm{DL}_K^{\tau}(V_i,\psi_i,D_i)$ *for every* $\tau \in \mathrm{Gal}(K_e/K)$.

Proof. 1. Let Δ be the homomorphism that maps $\mathrm{Iso}_{(V,\psi)}$ naturally into

$$\mathrm{diag}(\mathrm{Iso}_{(V,\psi)},\ldots,\mathrm{Iso}_{(V,\psi)}) \subseteq \mathrm{Iso}_{(V^s,\psi^s)}\,. \tag{3.7}$$

Since $\mathbb{Q} \subseteq D$, we have $M_s(\mathbb{Q}) \subseteq M_s(D)$. Directly from the definition of the twisted decomposable Lefschetz group, we get $\mathrm{DL}_K^{\tau}(V^s,\psi^s,M_s(D)) \cong \Delta(\mathrm{DL}_K^{\tau}(V,\psi,D)) \cong \mathrm{DL}_K^{\tau}(V,\psi,D)$.
2. The proof is very similar to the proof of 1, using the fact that $\prod_{i=1}^{s} \mathbb{Q} \subset \prod_{i=1}^{t} D_i$.
3. This follows immediately from 1 and 2. $\square$

Remark 3.6. Theorem 3.5 remains true if we replace $\mathrm{DL}_K^{\tau}(V',\psi',D')$ with $\mathrm{DL}_K^{\tau}(V',\psi',D')^{\circ}$ for all polarized Hodge structures V',ψ' and corresponding rings D' with Galois actions that appear in the theorem. Since we have $\mathrm{L}(V',\psi',D') = \mathrm{DL}_K^{\mathrm{id}}(V',\psi',D')^{\circ}$, the Lefschetz group satisfies properties 1–3 of Theorem 3.5.

Remark 3.7. Observe that we have

$$\mathrm{DL}_K(V,\psi,D) := \{g \in \mathrm{Iso}_{(V,\psi)} : \exists\, \tau \in G_K\, \forall\, \beta \in D \quad g\beta g^{-1} = \rho_e(\tau)(\beta)\,\}$$

Changing quantifiers we get another group scheme

$$\widetilde{DL}_K(V,\psi,D) := \{g \in \mathrm{Iso}_{(V,\psi)} : \forall\, \beta \in D\, \exists\, \tau \in G_K \quad g\beta g^{-1} = \rho_e(\tau)(\beta)\,\} \tag{3.8}$$

Observe that $\mathrm{DL}_K(V,\psi,D) \subseteq \widetilde{\mathrm{DL}}_K(V,\psi,D)$.

Remark 3.8. Observe that (2.18) implies that

$$\mathrm{H}(V,\psi) \subseteq \mathrm{DL}_K^{\mathrm{id}}(V,\psi,D) \subseteq \mathrm{DL}_K(V,\psi,D). \tag{3.9}$$

4. Hodge structures associated with l-adic representations

Let (V,ψ) be a rational pure polarized Hodge structure of weight $n \neq 0$. Put $V_l \cong V \otimes_{\mathbb{Q}} \mathbb{Q}_l$ and $\psi_l := \psi \otimes_{\mathbb{Q}} \mathbb{Q}_l$. Let $(V_l,\psi_l) := (V \otimes_{\mathbb{Q}} \mathbb{Q}_l, \psi \otimes_{\mathbb{Q}} \mathbb{Q}_l)$ and assume that the bilinear form $\psi_l : V_l \times V_l \to \mathbb{Q}_l(-n)$ is G_K-equivariant and the family of l-adic representations

$$\rho_l : G_K \to \mathrm{GIso}(V_l,\psi_l) \tag{4.1}$$

is of Hodge-Tate type and strictly compatible in the sense of Serre. We assume that outside of a finite set of primes of $\mathcal{O}_K$, for each v the complex absolute values of the eigenvalues of a Frobenius element at v are $q_v^{\frac{n}{2}}$.

The form ψ_l is $(-1)^n$-symmetric by the assumptions on the Hodge structure. Hence

$$\mathrm{GIso}(V_l,\psi_l) \;=\; \begin{cases} \mathrm{GO}(V_l,\psi_l) & \text{if} \quad n\,\text{even}; \\ \mathrm{GSp}(V_l,\psi_l) & \text{if} \quad n\,\text{odd}. \end{cases} \tag{4.2}$$

Let χ be the character defined in (2.2) and let $\chi_c : G_K \to \mathbb{Z}_l^{\times}$ be the cyclotomic character. Then by the G_K-equivariance of ψ_l we obtain:

$$\chi \circ \rho_l \;=\; \chi_c^{-n}. \tag{4.3}$$

Remark 4.1. For a representation ρ_l of Hodge-Tate type, the theorem of Bogomolov on homotheties (cf. [**Su**, Prop. 2.8]) applies, meaning that $\rho_l(G_K) \cap \mathbb{Q}_l^{\times}\,\mathrm{Id}_{V_l}$ is open in $\mathbb{Q}_l^{\times}\,\mathrm{Id}_{V_l}$. Moreover, Bogomolov proved [**Bog**, Théorème 1] that $\rho_l(G_K)$ is open in $G_{l,K}^{\mathrm{alg}}(\mathbb{Q}_l)$.

Remark 4.2. Strictly compatible families of l-adic representations of Hodge-Tate type arise naturally from étale cohomology. Indeed, if X/K is a proper scheme and $\overline{X} := X \otimes_K \overline{F}$ then $V_{l,\mathrm{et}}^i := H_{\mathrm{et}}^i(\overline{X},\, \mathbb{Q}_l)$ is potentially semistable for each G_{K_v}-representation for every $v|l$ (see [**Ts1**, Cor. 2.2.3], [**Ts2**]). Hence the representation

$$\rho_{l,\mathrm{et}}^i : G_K \to \mathrm{GL}(V_{l,\mathrm{et}}^i) \tag{4.4}$$

is of Hodge-Tate type (cf. [**Su**, p. 603]).

Definition 4.3. Let

$$G_{l,K}^{\mathrm{alg}} := G_{l,K}^{\mathrm{alg}}(V,\psi) \subset \mathrm{GIso}_{(V_l,\psi_l)} \tag{4.5}$$

be the Zariski closure of $\rho_l(G_K)$ in $\mathrm{GIso}_{(V_l,\psi_l)}$. Put:

$$\rho_l(G_K)_1 := \rho_l(G_K) \cap \mathrm{Iso}_{(V_l,\psi_l)}, \tag{4.6}$$

$$G_{l,K,1}^{\mathrm{alg}} := G_{l,K,1}^{\mathrm{alg}}(V,\psi) := G_{l,K}^{\mathrm{alg}} \cap \mathrm{Iso}_{(V_l,\psi_l)}\,. \tag{4.7}$$

By the theorem of Bogomolov on homotheties (see Remark 4.1), there is an exact sequence

$$1 \longrightarrow G_{l,K,1}^{\mathrm{alg}} \longrightarrow G_{l,K}^{\mathrm{alg}} \xrightarrow{\chi} \mathbb{G}_m \longrightarrow 1. \tag{4.8}$$

Remark 4.4. If ρ_l is semisimple, then $G_{l,K}^{\mathrm{alg}}$ is reductive; hence in this case the algebraic group $G_{l,K,1}^{\mathrm{alg}}$ is also reductive, by virtue of being the kernel of a homomorphism from a reductive group to a torus.

Naturally $\rho_l(G_K)_1 \subseteq G_{l,K,1}^{\mathrm{alg}}$. Let $K \subseteq L \subset \overline{F}$ be a tower of extensions with L/K finite. Consider the following commutative diagram with left and middle vertical arrows injective:

$$\begin{tikzcd}
1 \arrow[r] & G_{l,K,1}^{\mathrm{alg}} \arrow[r] & G_{l,K}^{\mathrm{alg}} \arrow[r, "\chi"] & \mathbb{G}_m \arrow[r] & 1 \\
1 \arrow[r] & G_{l,L,1}^{\mathrm{alg}} \arrow[r] \arrow[u] & G_{l,L}^{\mathrm{alg}} \arrow[r, "\chi"] \arrow[u] & \mathbb{G}_m \arrow[r] \arrow[u, "="'] & 1
\end{tikzcd} \tag{4.9}$$

It is clear that $G_{l,K,1}^{\mathrm{alg}} \cap G_{l,L}^{\mathrm{alg}} = G_{l,L,1}^{\mathrm{alg}}$. If L/K is Galois, then it follows from the diagram (4.9) that there is a monomorphism:

$$j_{L/K}\ :\ \ \rho_l(G_K)_1\,/\,\rho_l(G_L)_1 \ \hookrightarrow\ \rho_l(G_K)\,/\,\rho_l(G_L). \tag{4.10}$$

Proposition 4.5. *Let $K \subset L \subset M$ with M/K and L/K Galois. The map $j_{M/K}$ is an isomorphism if and only if $j_{M/L}$ and $j_{L/K}$ are isomorphisms.*

We observe that for any finite Galois extension L/K the natural map is an epimorphism $\mathrm{Zar}_{L/K} := \mathrm{Zar}_{l,\,L/K}$:

$$\mathrm{Zar}_{L/K}\ :\ \ \rho_l(G_K)\,/\,\rho_l(G_L) \twoheadrightarrow G_{l,K}^{\mathrm{alg}}\,/\,G_{l,L}^{\mathrm{alg}}\,. \tag{4.11}$$

The proofs of the following three results: Theorem 4.6, Proposition 4.7, Theorem 4.8, are similar to the proofs of [**BK**, Theorem 3.1, Proposition 3.2, Theorem 3.3]. Theorem 4.8 is a generalization of the result of Serre [**Se3**, §8.3.4]. As usual, for an algebraic group G we put $\pi_0(G) := G/G^\circ$.

Theorem 4.6. *Let $K \subseteq L \subset \overline{F}$ with L/K finite Galois. The following natural map is an isomorphism of finite groups:*

$$i_{L/K}\ :\ G_{l,K,1}^{\mathrm{alg}}/G_{l,L,1}^{\mathrm{alg}} \ \xrightarrow{\cong}\ G_{l,K}^{\mathrm{alg}}/G_{l,L}^{\mathrm{alg}}. \tag{4.12}$$

In particular there are the following equalities:

$$(G_{l,L}^{\mathrm{alg}})^\circ = (G_{l,K}^{\mathrm{alg}})^\circ \quad \text{and} \quad (G_{l,L,1}^{\mathrm{alg}})^\circ = (G_{l,K,1}^{\mathrm{alg}})^\circ. \tag{4.13}$$

Proof. It is clear that $G_{l,L}^{\mathrm{alg}} \lhd G_{l,K}^{\mathrm{alg}}$ and $G_{l,L,1}^{\mathrm{alg}} \lhd G_{l,K,1}^{\mathrm{alg}}$. On the other hand, there is a surjective homomorphism $\rho_l(G_K)/\rho_l(G_L) \to G_{l,K}^{\mathrm{alg}}/G_{l,L}^{\mathrm{alg}}$, so $G_{l,L}^{\mathrm{alg}}$ is a subgroup of $G_{l,K}^{\mathrm{alg}}$ of finite index. In particular, $(G_{l,L}^{\mathrm{alg}})^\circ = (G_{l,K}^{\mathrm{alg}})^\circ$.

The following commutative diagram has exact rows. The left and the middle columns are also exact cf. (4.8).

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & G_{l,L,1}^{\mathrm{alg}} \arrow[r] \arrow[d] & G_{l,K,1}^{\mathrm{alg}} \arrow[r] \arrow[d] & G_{l,K,1}^{\mathrm{alg}}/G_{l,L,1}^{\mathrm{alg}} \arrow[r] \arrow[d, "\cong"', "i_L"] & 1 \\
1 \arrow[r] & G_{l,L}^{\mathrm{alg}} \arrow[r] \arrow[d, "\chi"] & G_{l,K}^{\mathrm{alg}} \arrow[r] \arrow[d, "\chi"] & G_{l,K}^{\mathrm{alg}}/G_{l,L}^{\mathrm{alg}} \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r, "="] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d] & 1 & \\
 & 1 & 1 & &
\end{tikzcd}$$

Then a diagram chase (as in the snake lemma) shows that the third column is also exact, so the map i_L is an isomorphism. Hence it is clear that $(G_{l,L,1}^{\mathrm{alg}})^\circ = (G_{l,K,1}^{\mathrm{alg}})^\circ$. ☐

Proposition 4.7. *Let the weight of the Hodge structure be the odd integer $n = 2m+1$. There is a finite Galois extension L_0/K such that $G_{l,L_0}^{\mathrm{alg}} = (G_{l,K}^{\mathrm{alg}})^\circ$ and $G_{l,L_0,1}^{\mathrm{alg}} = (G_{l,K,1}^{\mathrm{alg}})^\circ$.*

Proof. Since the subscheme $(G_{l,K}^{\mathrm{alg}})^\circ$ is open and closed in $G_{l,K}^{\mathrm{alg}}$ and ρ_l is continuous, we can find a finite Galois extension L_0/K such that $\rho_l(G_{L_0}) \subset (G_{l,K}^{\mathrm{alg}})^\circ$. Hence $G_{l,L_0}^{\mathrm{alg}} \subseteq (G_{l,K}^{\mathrm{alg}})^\circ$. Since we already have the reverse inclusion, we obtain the first desired equality.

Consider the restriction of the l-adic representation to the base field L_0. Using the Hodge-Tate property of V_l, after taking $\mathbb{C}$ points in the exact sequence (4.8) one can apply the homomorphism h [**Se3**, p. 114] defined by Serre to get the homomorphism:

$$h : \mathbb{G}_m(\mathbb{C}) \to G_{l,L_0}^{\mathrm{alg}}(\mathbb{C})$$

such that for all $x \in \mathbb{G}_m(\mathbb{C})$, $h(x)$ acts by multiplication by x^p on the subspace $V^{p,n-p}$. One checks that $\chi(h(x)) = x^n$ for every $x \in \mathbb{G}_m(\mathbb{C})$ (see the diagram preceding Definition 2.6). Let

$$w : \mathbb{G}_m(\mathbb{C}) \to G_{l,L_0}^{\mathrm{alg}}(\mathbb{C})$$

$$w(x) = x\,\mathrm{Id}_{V_{\mathbb{C}}}$$

be the diagonal homomorphism; this is well-defined thanks to Remark 4.1. We know (Remark 2.4) that $\chi(w(x)) = x^2$ for every $x \in \mathbb{G}_m(\mathbb{C})$. Hence the homomorphism

$$s : \mathbb{G}_m(\mathbb{C}) \to G^{\mathrm{alg}}_{l,L_0}(\mathbb{C})$$
$$s(x) := h(x)w(x)^{-m}$$

is a splitting of χ in the following exact sequence:

$$1 \longrightarrow G^{\mathrm{alg}}_{l,L_0,1}(\mathbb{C}) \longrightarrow G^{\mathrm{alg}}_{l,L_0}(\mathbb{C}) \xrightarrow{\chi} \mathbb{C}^\times \longrightarrow 1.$$

Observe that $G^{\mathrm{alg}}_{l,L_0}(\mathbb{C})$ is a connected Lie group. Take any two points g_0 and g_1 in $G^{\mathrm{alg}}_{l,L_0,1}(\mathbb{C})$. There is a path $\alpha(t) \in G^{\mathrm{alg}}_{l,L_0}(\mathbb{C})$ connecting g_0 and g_1, i.e., $\alpha(0) = g_0$ and $\alpha(1) = g_1$. Define a new path

$$\beta(t) := s(\chi(\alpha(t)))^{-1}\alpha(t) \in G^{\mathrm{alg}}_{l,K_0}(\mathbb{C})$$

Observe that: $\chi(\beta(t)) := \chi(s(\chi(\alpha(t)))^{-1})\chi(\alpha(t)) = \chi(\alpha(t))^{-1}\chi(\alpha(t)) = 1$. We easily check that $\beta(0) = g_0$ and $\beta(1) = g_1$. Hence $\beta(t) \in G^{\mathrm{alg}}_{l,K_0,1}(\mathbb{C})$ connects g_0 and g_1. It follows that $G^{\mathrm{alg}}_{l,L_0,1}$ is connected, hence $G^{\mathrm{alg}}_{l,L_0,1} = (G^{\mathrm{alg}}_{l,K,1})^\circ$. □

THEOREM 4.8. *Let n be odd. The following natural map is an isomorphism:*

$$i_{CC} \ : \ \pi_0(G^{\mathrm{alg}}_{l,K,1}) \xrightarrow{\cong} \pi_0(G^{\mathrm{alg}}_{l,K}).$$

PROOF. Choose L_0 as in Proposition 4.7. Put $L := L_0$ in the diagram of the proof of Theorem 4.6. Then $i_{CC} = i_{L_0}$, which is an isomorphism by Theorem 4.6. □

Remark 4.9. The natural continuous action by left translation:

$$G_K \times \pi_0(G^{\mathrm{alg}}_{l,K}) \to \pi_0(G^{\mathrm{alg}}_{l,K}) \tag{4.14}$$

and Theorem 4.8 give the following continuous action by left translation:

$$G_K \times \pi_0(G^{\mathrm{alg}}_{l,K,1}) \to \pi_0(G^{\mathrm{alg}}_{l,K,1}). \tag{4.15}$$

5. Algebraic Sato-Tate conjecture

In this chapter we assume that the Hodge structure (V, ψ), the ring D and the family of l-adic representations $\rho_l \ : \ G_K \to \mathrm{GIso}(V_l, \psi_l)$ satisfy all the properties assumed in chapters 2–4. We also assume hereafter that n is odd; the case where n is even requires some modifications to the definitions, which we will discuss elsewhere.

One of the main objectives of this paper is the investigation of the following conjecture:

CONJECTURE 5.1. *(Algebraic Sato-Tate conjecture)*
(a) *For every finite extension K/F and for every l, there exist a natural-in-K reductive algebraic group $\mathrm{AST}_K(V, \psi) \subset \mathrm{Iso}_{(V,\psi)}$ over $\mathbb{Q}$ and a natural-in-K monomorphism of group schemes:*

$$\mathrm{ast}_{l,K} \ : \ G^{\mathrm{alg}}_{l,K,1} \hookrightarrow \mathrm{AST}_K(V, \psi)_{\mathbb{Q}_l}. \tag{5.1}$$

(b) *The map (5.1) is an isomorphism:*

$$\mathrm{ast}_{l,K} \ : \ G^{\mathrm{alg}}_{l,K,1} \xrightarrow{\cong} \mathrm{AST}_K(V, \psi)_{\mathbb{Q}_l}. \tag{5.2}$$

Remark 5.2. We say that an algebraic group is *reductive* if its identity connected component is reductive.

Remark 5.3. The requirement that $\mathrm{AST}_K(V,\psi)$ and (5.1) are natural in K means that for any finite extension L/K there is a natural monomorphism of group schemes:

$$\mathrm{AST}_L(V,\psi) \hookrightarrow \mathrm{AST}_K(V,\psi)$$

making the following diagram commute:

$$\begin{tikzcd} G_{l,K,1}^{\mathrm{alg}} \arrow[r, "\mathrm{ast}_{l,K}"] & \mathrm{AST}_K(V,\psi)_{\mathbb{Q}_l} \\ G_{l,L,1}^{\mathrm{alg}} \arrow[u] \arrow[r, "\mathrm{ast}_{l,L}"] & \mathrm{AST}_L(V,\psi)_{\mathbb{Q}_l} \arrow[u] \end{tikzcd}$$

Definition 5.4. The group $\mathrm{AST}_K(V,\psi)$ is called the *algebraic Sato-Tate group*. A maximal compact subgroup of $\mathrm{AST}_K(V,\psi)(\mathbb{C})$ is called the *Sato-Tate group* and is denoted $\mathrm{ST}_K(V,\psi)$.

Remark 5.5. We will make the following abbreviations: $\mathrm{AST}_K := \mathrm{AST}_K(V,\psi)$ and $\mathrm{ST}_K := \mathrm{ST}_K(V,\psi)$, whenever they do not lead to a notation conflict.

Remark 5.6. When the Hodge structure (V,ψ) comes from the cohomology of a smooth, projective variety over K, then Conjecture 5.1 is closely related to the Tate conjecture.

Choose a suitable field embedding $\mathbb{Q}_l \to \mathbb{C}$ and put $G_{l,K,1_{\mathbb{C}}}^{\mathrm{alg}} := G_{l,K,1}^{\mathrm{alg}} \otimes_{\mathbb{Q}_l} \mathbb{C}$. Naturally we have $\pi_0(G_{l,K,1}^{\mathrm{alg}}) \cong \pi_0(G_{l,K,1_{\mathbb{C}}}^{\mathrm{alg}})$. By Theorem 4.8 and an argument similar to the proof of [**FKRS**, Lemma 2.8], we have the following.

Proposition 5.7. *Assume that the algebraic Sato-Tate conjecture (Conjecture 5.1) holds. Then there are natural isomorphisms*

$$\pi_0(G_{l,K,1}^{\mathrm{alg}}) \;\cong\; \pi_0(\mathrm{AST}_K(V,\psi)) \;\cong\; \pi_0(\mathrm{ST}_K(V,\psi)). \tag{5.3}$$

Remark 5.8. Assume that the algebraic Sato-Tate conjecture (Conjecture 5.1) holds. Then obviously the Sato-Tate group $\mathrm{ST}_K(V,\psi)$ is independent of l. Take a prime v in $\mathcal{O}_K$ and take a Frobenius element Fr_v in G_K. Following [**Se3**, §8.3.3] (cf. [**FKRS**, Def. 2.9]) one can make the following construction. Let s_v be the semisimple part in $\mathrm{SL}_V(\mathbb{C})$ of the element

$$q_v^{-\frac{n}{2}}\rho_l(\mathrm{Fr}_v) \;\in\; G_{l,K,1}^{\mathrm{alg}}(\mathbb{C}) \;\cong\; \mathrm{AST}_K(V,\psi)(\mathbb{C}) \;\subset\; \mathrm{Iso}_{(V,\psi)}(\mathbb{C}) \;\subset\; \mathrm{SL}_V(\mathbb{C});$$

since the family (ρ_l) is strictly compatible, s_v is independent of l. By [**Hu**, Theorem 15.3 (c) p. 99], the semisimple part of $q_v^{-\frac{n}{2}}\rho_l(\mathrm{Fr}_v)$ considered in $\mathrm{Iso}_{(V,\psi)}(\mathbb{C})$ and in $\mathrm{AST}_K(V,\psi)(\mathbb{C})$ is again s_v, and so is again independent of l. Hence $\mathrm{conj}(s_v)$ in $\mathrm{AST}_K(V,\psi)(\mathbb{C})$ is independent of l. Obviously $\mathrm{conj}(s_v) \subset \mathrm{AST}_K(V,\psi)(\mathbb{C})$ is independent of the choice of a Frobenius element Fr_v over v and contains the semisimple parts of all the elements of $\mathrm{conj}(q_v^{-\frac{n}{2}}\rho_l(\mathrm{Fr}_v))$ in $\mathrm{AST}_K(V,\psi)(\mathbb{C})$. Moreover, the elements in $\mathrm{conj}(s_v)$ have eigenvalues of complex absolute value 1 by our assumptions, so there is some conjugate of s_v contained in $\mathrm{ST}_K(V,\psi)$. This allows us to make sense of the following conjecture.

CONJECTURE 5.9. *(Sato-Tate conjecture) The conjugacy classes* $\mathrm{conj}(s_v)$ *in* $\mathrm{ST}_K(V,\psi)$ *are equidistributed in* $\mathrm{conj}(\mathrm{ST}_K(V,\psi))$ *with respect to the measure induced by the Haar measure of* $\mathrm{ST}_K(V,\psi)$.

Remark 5.10. If we are only interested in the isomorphism (5.3) for a fixed l, then it is enough to assume the existence of $\mathrm{AST}_K(V,\psi)$, as in Conjecture 5.1, and the existence of $\mathrm{ast}_{l,K}$ which is an isomorphism for this particular l.

We now use the twisted Lefschetz group to obtain an upper bound on the algebraic Sato-Tate group.

Remark 5.11. We will assume in this and the next three chapters that the induced action of D on V_l is G_K-equivariant. In other words, $\forall \beta \in D$, $\forall v_l \in V_l$ and $\forall \sigma \in G_K$:

$$\rho_l(\sigma)(\beta\, v_l) = \sigma(\beta)\, \rho_l(\sigma)(v_l). \tag{5.4}$$

This immediately gives:

$$\rho_l(\sigma)\beta\rho_l(\sigma^{-1})(v_l) = \sigma(\beta)(v_l), \tag{5.5}$$

$$\rho_l(G_K)_1 \subseteq \mathrm{DL}_K(V,\psi,D)(\mathbb{Q}_l). \tag{5.6}$$

We will observe, by (5.8) and (5.12) below, that:

$$G_{l,K,1}^{\mathrm{alg}} \subseteq \mathrm{DL}_K(V,\psi,D)_{\mathbb{Q}_l}, \tag{5.7}$$

We are interested in finding polarized Hodge structures (V,ψ) and rings D for which $G_{l,K,1}^{\mathrm{alg}} = \mathrm{DL}_K(V,\psi,D)_{\mathbb{Q}_l}$ for each l. In such cases $\mathrm{AST}_K(V,\psi) = \mathrm{DL}_K(V,\psi,D)$. We explain in this chapter that the equality $G_{l,K,1}^{\mathrm{alg}} = \mathrm{DL}_K(V,\psi,D)_{\mathbb{Q}_l}$ is equivalent to $G_{l,K_e,1}^{\mathrm{alg}} = \mathrm{DL}_{K_e}(V,\psi,D)_{\mathbb{Q}_l}$.

DEFINITION 5.12. Put:

$$(G_{l,K}^{\mathrm{alg}})^\tau := \{g \in G_{l,K}^{\mathrm{alg}} :\ g\beta g^{-1} = \rho_e(\tau)(\beta) \quad \forall\, \beta \in D\},$$
$$(G_{l,K,1}^{\mathrm{alg}})^\tau := (G_{l,K}^{\mathrm{alg}})^\tau \cap G_{l,K,1}^{\mathrm{alg}}.$$

Observe that

$$(G_{l,K,1}^{\mathrm{alg}})^\tau \subseteq \mathrm{DL}_K^\tau(V,\psi,D)_{\mathbb{Q}_l}. \tag{5.8}$$

Remark 5.13. Let $\tilde\tau \in G_K$ be a lift of $\tau \in \mathrm{Gal}(K_e/K)$. The coset $\tilde\tau\, G_{K_e}$ does not depend on the lift. The Zariski closure of $\rho_l(\tilde\tau\, G_{K_e}) = \rho_l(\tilde\tau)\, \rho_l(G_{K_e})$ in $\mathrm{GIso}_{(V_l,\psi_l)}$ is $\rho_l(\tilde\tau)\, G_{l,K_e}^{\mathrm{alg}}$. Since $\rho_l(\tilde\tau)\, \rho_l(G_{K_e}) \subset (G_{l,K}^{\mathrm{alg}})^\tau$ then $\rho_l(\tilde\tau)\, G_{l,K_e}^{\mathrm{alg}} \subset (G_{l,K}^{\mathrm{alg}})^\tau$. Because:

$$\rho_l(G_K) = \bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} \rho_l(\tilde\tau)\, \rho_l(G_{K_e}), \tag{5.9}$$

then

$$G_{l,K}^{\mathrm{alg}} = \bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} \rho_l(\tilde\tau)\, G_{l,K_e}^{\mathrm{alg}}. \tag{5.10}$$

This implies the following equalities:

$$G_{l,K}^{\mathrm{alg}} = \bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} (G_{l,K}^{\mathrm{alg}})^{\tau}, \tag{5.11}$$

$$G_{l,K,1}^{\mathrm{alg}} = \bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} (G_{l,K,1}^{\mathrm{alg}})^{\tau}. \tag{5.12}$$

Now we observe that $\rho_l(\tilde{\tau})\, G_{l,K_e}^{\mathrm{alg}} = (G_{l,K}^{\mathrm{alg}})^{\tau}$ for all τ. This implies the equality $(G_{l,K,1}^{\mathrm{alg}})^{\mathrm{id}} = G_{l,K_e,1}^{\mathrm{alg}}$ and the following natural isomorphism:

$$G_{l,K}^{\mathrm{alg}}/(G_{l,K}^{\mathrm{alg}})^{\mathrm{id}} \cong \mathrm{Gal}(K_e/K). \tag{5.13}$$

Since $\mathrm{DL}_K^{\mathrm{id}}(V,\psi,D) = \mathrm{DL}_{K_e}(V,\psi,D) = \mathrm{DL}_{\overline{F}}(V,\psi,D)$, we get

$$G_{l,K_e,1}^{\mathrm{alg}} \subseteq \mathrm{DL}_{K_e}(V,\psi,D)_{\mathbb{Q}_l}. \tag{5.14}$$

Hence by (5.8), (5.13) and Theorem 4.8 there are natural isomorphisms:

$$G_{l,K,1}^{\mathrm{alg}}/(G_{l,K,1}^{\mathrm{alg}})^{\mathrm{id}} \;\cong\; \mathrm{DL}_K(V,\psi,D)/\,\mathrm{DL}_K^{\mathrm{id}}(V,\psi,D) \;\cong\; \mathrm{Gal}(K_e/K). \tag{5.15}$$

Theorem 5.14. *The following equalities are equivalent:*

$$G_{l,K_e,1}^{\mathrm{alg}} = \mathrm{DL}_{K_e}(V,\psi,D)_{\mathbb{Q}_l}. \tag{5.16}$$

$$G_{l,K,1}^{\mathrm{alg}} = \mathrm{DL}_K(V,\psi,D)_{\mathbb{Q}_l}. \tag{5.17}$$

Let L/K be a finite extension such that $L \subset \overline{F}$. The following equalities are equivalent:

$$G_{l,L_e,1}^{\mathrm{alg}} = \mathrm{DL}_{L_e}(V,\psi,D)_{\mathbb{Q}_l}. \tag{5.18}$$

$$G_{l,L,1}^{\mathrm{alg}} = \mathrm{DL}_L(V,\psi,D)_{\mathbb{Q}_l}. \tag{5.19}$$

Moreover equalities (5.18) *and* (5.19) *imply equalities* (5.16) *and* (5.17).

Proof. The equivalence of (5.16) and (5.17) follows from (5.15). Changing base to an extension L/K, the equivalence of (5.18) and (5.19) also follows from (5.15). Observe that $\mathrm{Ker}(\rho_e|G_L) \subset \mathrm{Ker}\,\rho_e$. Hence $K_e \subset L_e$. It follows that $\mathrm{DL}_{K_e}(V,\psi,D) = \mathrm{DL}_{L_e}(V,\psi,D)$ and $G_{l,L_e,1}^{\mathrm{alg}} \subset G_{l,K_e,1}^{\mathrm{alg}}$. Hence (5.14) and (5.18) imply (5.16). $\square$

6. Connected components of AST_K and ST_K

Remark 6.1. Consider the continuous homomorphism

$$\epsilon_{l,K} \,:\, G_K \to G_{l,K}^{\mathrm{alg}}(\mathbb{Q}_l). \tag{6.1}$$

Since $\rho_l(G_K)$ is Zariski dense in $G_{l,K}^{\mathrm{alg}}$, this map induces the continuous epimorphism:

$$\tilde{\epsilon}_{l,K} \,:\, G_K \to \pi_0(G_{l,K}^{\mathrm{alg}}). \tag{6.2}$$

Since $(G_{l,K}^{\mathrm{alg}})^{\circ}$ is open in $G_{l,K}^{\mathrm{alg}}$, we get:

$$\epsilon_{l,K}^{-1}((G_{l,K}^{\mathrm{alg}})^{\circ}(\mathbb{Q}_l)) \;=\; \mathrm{Ker}\,\tilde{\epsilon}_{l,K} \;=\; G_{K_0} \tag{6.3}$$

for some finite Galois extension K_0/K. From Proposition 4.7 and Theorem 4.8 it follows that K_0/K is the minimal extension such that $G^{\mathrm{alg}}_{l,K_0} = (G^{\mathrm{alg}}_{l,K})^\circ$ and $G^{\mathrm{alg}}_{l,K_0,1} = (G^{\mathrm{alg}}_{l,K,1})^\circ$. In principle, K_0 may depend on l; in Proposition 6.5 below, we will give conditions for the independence of K_0 from l. These conditions are satisfied in the case of abelian varieties; see Remark 6.13.

Let $\tilde\sigma \in G_K$ be a lift of $\sigma \in \mathrm{Gal}(K_0/K)$. The coset $\tilde\sigma\, G_{K_0}$ does not depend on the lift. By the definition of K_0, there is an obvious isomorphism:

$$G^{\mathrm{alg}}_{l,K}/(G^{\mathrm{alg}}_{l,K})^\circ \cong \mathrm{Gal}(K_0/K). \tag{6.4}$$

Also, the Zariski closure of $\rho_l(\tilde\sigma\, G_{K_0}) = \rho_l(\tilde\sigma)\,\rho_l(G_{K_0})$ in $\mathrm{GIso}_{(V_l,\psi_l)}$ is $\rho_l(\tilde\tau)\, G^{\mathrm{alg}}_{l,K_0}$. Because

$$\rho_l(G_K) = \bigsqcup_{\sigma\in\mathrm{Gal}(K_0/K)} \rho_l(\tilde\sigma)\,\rho_l(G_{K_0}), \tag{6.5}$$

by the definition of K_0 we have:

$$G^{\mathrm{alg}}_{l,K} = \bigsqcup_{\sigma\in\mathrm{Gal}(K_0/K)} \rho_l(\tilde\sigma)\, G^{\mathrm{alg}}_{l,K_0}. \tag{6.6}$$

Remark 6.2. Let $H_{l,K,1} := \rho_l^{-1}(\rho_l(G_K)_1)$ and $K_1 := \overline{K}^{H_{l,K,1}}$. Observe that:

$$\epsilon^{-1}_{l,K}((G^{\mathrm{alg}}_{l,K,1})^\circ(\mathbb{Q}_l)) = \epsilon^{-1}_{l,K}(G^{\mathrm{alg}}_{l,K_0,1}(\mathbb{Q}_l)) = \epsilon^{-1}_{l,K}((G^{\mathrm{alg}}_{l,K_0}\cap \mathrm{Iso}_{(V_l,\psi_l)})(\mathbb{Q}_l)) =$$
$$= \epsilon^{-1}_{l,K}(G^{\mathrm{alg}}_{l,K_0}(\mathbb{Q}_l))\,\cap\,\epsilon^{-1}_{l,K}(\mathrm{Iso}_{(V_l,\psi_l)}(\mathbb{Q}_l)) = G_{K_0}\cap G_{K_1} = G_{K_0K_1}.$$

Remark 6.3. We observe that $K\subset K_e\subset K_0$.

PROPOSITION 6.4. *Assume Conjecture 5.1* (a) *and assume that* $\mathrm{ast}_{l,K}$ *and* ast_{l,K_0} *are isomorphisms for a fixed* l. *Let* L/K_0 *be a finite Galois extension. Then:*
(1) $\mathrm{AST}_{K_0} = (\mathrm{AST}_K)^\circ$.
(2) $\mathrm{ST}_{K_0} = (\mathrm{ST}_K)^\circ$ *up to conjugation in* $\mathrm{AST}_K(\mathbb{C})$.
(3) $\mathrm{AST}_{K_0} = \mathrm{AST}_L$.
(4) $\mathrm{ST}_{K_0} = \mathrm{ST}_L$ *up to conjugation in* $\mathrm{AST}_{K_0}(\mathbb{C})$.

PROOF. Consider the following commutative diagram. The bottom row is exact. The right vertical arrow is an isomorphism by (5.3) of Proposition 5.7 (cf. Remark 5.10).

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{AST}_{K_0,\mathbb{Q}_l} \arrow[r] & \mathrm{AST}_{K,\mathbb{Q}_l} \arrow[r] & \pi_0(\mathrm{AST}_{K,\mathbb{Q}_l}) \arrow[r] & 1 \\ 1 \arrow[r] & G^{\mathrm{alg}}_{l,K_0,1} \arrow[r] \arrow[u, "\mathrm{ast}_{l,K_0}", "\cong"'] & G^{\mathrm{alg}}_{l,K,1} \arrow[r] \arrow[u, "\mathrm{ast}_{l,K}", "\cong"'] & \pi_0(G^{\mathrm{alg}}_{l,K,1}) \arrow[r] \arrow[u, "\cong"'] & 1 \end{tikzcd} \tag{6.7}$$

Since $\mathrm{AST}_{K_0,\mathbb{Q}_l}$ is connected (since it is isomorphic to $G^{\mathrm{alg}}_{l,K_0,1}$), the exactness of the top row in (6.7) implies $\mathrm{AST}_{K_0,\mathbb{Q}_l} = (\mathrm{AST}_{K,\mathbb{Q}_l})^\circ$ and in particular $\mathrm{AST}_{K_0} = (\mathrm{AST}_K)^\circ$. By Proposition 5.7 we obtain $\pi_0(\mathrm{ST}_K) = \pi_0(\mathrm{AST}_K)$ and $\pi_0(\mathrm{ST}_{K_0}) = \pi_0(\mathrm{AST}_{K_0}) = 1$. Hence by (1) we have $\mathrm{ST}_{K_0}\subset(\mathrm{ST}_K)^\circ$ up to conjugation in $\mathrm{AST}_K(\mathbb{C})$ because ST_{K_0} is connected and compact and ST_K maximal compact in $\mathrm{AST}_K(\mathbb{C})$. On the other hand $(\mathrm{ST}_K)^\circ\subset(\mathrm{AST}_K)^\circ(\mathbb{C}) = \mathrm{AST}_{K_0}(\mathbb{C})$ by $\pi_0(\mathrm{ST}_K) = \pi_0(\mathrm{AST}_K)$ and by (1). Hence $(\mathrm{ST}_K)^\circ\subset\mathrm{ST}_{K_0}$ up to conjugation in $\mathrm{AST}_{K_0}(\mathbb{C})$ because ST_{K_0} is maximal compact in $\mathrm{AST}_{K_0}(\mathbb{C})$. Hence (2) follows. To prove (3)

observe that $G^{\mathrm{alg}}_{l,K_0,1} = G^{\mathrm{alg}}_{l,L,1}$ because $G^{\mathrm{alg}}_{l,L,1}$ is a normal subgroup of finite index in $G^{\mathrm{alg}}_{l,K_0,1}$ and $G^{\mathrm{alg}}_{l,K_0,1}$ is connected. Then (3) follows from the following commutative diagram:

$$\begin{tikzcd} G^{\mathrm{alg}}_{l,K_0,1} \arrow[r, "\mathrm{ast}_{l,K_0}", "\cong"'] & \mathrm{AST}_{K_0,\,\mathbb{Q}_l} \\ G^{\mathrm{alg}}_{l,L,1} \arrow[u, "="] \arrow[r, "\mathrm{ast}_{l,L}"] & \mathrm{AST}_{L,\,\mathbb{Q}_l} \arrow[u] \end{tikzcd}$$

and (3) implies (4) directly. □

Proposition 6.5. *Assume that Conjecture 5.1 holds for K and K_0. Then the field K_0 is independent of l.*

Proof. Assume that the corresponding equality to (6.3) holds for l' and K_0'. Hence by Remark 5.3, the assumptions and Proposition 6.4 we have $(\mathrm{AST}_K)^\circ \cong \mathrm{AST}_{K_0} \cong \mathrm{AST}_{K_0'}$. Then from continuity of the maps $\epsilon_{l',K}$ and $\tilde{\epsilon}_{l',K}$ we find out that $K_0' \subset K_0$. By symmetry, from continuity of the maps $\epsilon_{l,K}$ and $\tilde{\epsilon}_{l,K}$ we obtain $K_0 \subset K_0'$. □

Remark 6.6. Let $C \in \mathbb{N}$ be fixed. Then Proposition 6.5 has the following version for all $l \geq C$.

Proposition 6.7. *Assume that for every $l \geq C$ the homomorphisms $\mathrm{ast}_{l,K}$ and ast_{l,K_0} are isomorphisms. Then the field K_0 is independent of $l \geq C$.*

The surjectivity of (4.10) is a subtle point in the computation of Sato-Tate groups. Below we find conditions for the surjectivity. Let L/K be a finite Galois extension. Consider the following commutative diagram where $\mathrm{Zar}_{L/K} := \mathrm{Zar}_{l,\,L/K}$ and $\mathrm{Zar}_{L/K,\,1} := \mathrm{Zar}_{l,\,L/K,\,1}$.

$$\begin{tikzcd} \rho_l(G_K)/\rho_l(G_L) \arrow[r, two heads, "\mathrm{Zar}_{L/K}"] & G^{\mathrm{alg}}_{l,K}/G^{\mathrm{alg}}_{l,L} \\ \rho_l(G_K)_1/\rho_l(G_L)_1 \arrow[u, "j_{L/K}"] \arrow[r, "\mathrm{Zar}_{L/K,\,1}"] & G^{\mathrm{alg}}_{l,K,1}/G^{\mathrm{alg}}_{l,L,1} \arrow[u, "i_{L/K}", "\cong"'] \end{tikzcd} \tag{6.8}$$

We put

$$\bar{l} \;=\; \begin{cases} l & \text{if} \quad l > 2 \\ 8 & \text{if} \quad l = 2. \end{cases}$$

Let $K(\mu_{\bar{l}}^{\otimes n}) := \bar{K}^{\mathrm{Ker}\widetilde{\chi_c^n}}$, where $\widetilde{\chi_c^n} : G_K \to \mathrm{Aut}(\mu_{\bar{l}}^{\otimes n})$ is the n-th power of the cyclotomic character $\mod \bar{l}$.

Lemma 6.8. *Let L/K be a finite Galois extension. Assume that:*

(1) $L \cap K(\mu_{\bar{l}}^{\otimes\, n}) = K$;
(2) $1 + l\mathbb{Z}_l \,\mathrm{Id}_{V_l} \subset \rho_l(G_K)$;
(3) $\mathrm{Zar}_{L/K}$ *is an isomorphism.*

Then the maps $j_{L/K}$ and $\mathrm{Zar}_{L/K,\,1}$ are isomorphisms.

Proof. By assumption (3), the upper horizontal arrow in (6.8) is an isomorphism. The left vertical arrow (see (4.10)) is a monomorphism, and by Theorem 4.6 the right vertical arrow is an isomorphism. To show the theorem, it is enough to prove that the bottom horizontal arrow $\mathrm{Zar}_{L/K,\,1}$ is an epimorphism. For each $\sigma \in \mathrm{Gal}(L/K)$ we can choose, by assumption (1), a lift $\tilde{\sigma} \in \mathrm{Gal}(\overline{F}/K)$ such that

$$\tilde{\sigma}\,|_{K(\mu_{\overline{l}}^{\otimes\, n})} = \mathrm{Id}_{K(\mu_{\overline{l}}^{\otimes\, n})}.$$

Recall the natural exact sequence:

$$1 \to \mathrm{Iso}_{(V_l,\psi_l)} \to \mathrm{GIso}_{(V_l,\psi_l)} \xrightarrow{\chi} \mathbb{G}_m \to 1. \tag{6.9}$$

Since $\rho_l(G_K) \subset G_{l,K}^{\mathrm{alg}}(\mathbb{Q}_l) \subset \mathrm{GIso}_{(V_l,\psi_l)}(\mathbb{Q}_l)$, the choice of the lift $\tilde{\sigma}$ and the equality (4.3) give $\chi(\rho_l(\tilde{\sigma})) \in 1 + \bar{l}\,\mathbb{Z}_l \subset \mathbb{G}_m(\mathbb{Q}_l)$. Hence $\sqrt{\chi(\rho_l(\tilde{\sigma}))} \in 1 + l\mathbb{Z}_l$ because $(1 + l\mathbb{Z}_l)^2 = 1 + \bar{l}\,\mathbb{Z}_l$. By assumption (2), there exists $\tilde{\gamma} \in G_K$ such that $\rho_l(\tilde{\gamma}) = \sqrt{\chi(\rho_l(\tilde{\sigma}))}\,\mathrm{Id}_{V_l}$. By Remark 2.4, we have $\chi(\alpha\,\mathrm{Id}_{V_l}) = \alpha^2$ for any $\alpha \in \mathbb{Q}_l^{\times}$. Hence:

$$\chi(\rho_l(\tilde{\sigma}\tilde{\gamma}^{-1})) = \chi(\rho_l(\tilde{\sigma}))\chi(\rho_l(\tilde{\gamma}))^{-1} = 1. \tag{6.10}$$

It follows that $\rho_l(\tilde{\sigma}\tilde{\gamma}^{-1}) \in \rho_l(G_K)_1$. Since:

$$\rho_l(G_K) \;=\; \bigcup_{\sigma\in \mathrm{Gal}(L/K)} \rho_l(\tilde{\sigma})\,\rho_l(G_L) \;=\; {\bigsqcup}'_{\sigma\in \mathrm{Gal}(L/K)}\; \rho_l(\tilde{\sigma})\,\rho_l(G_L) \tag{6.11}$$

then by assumption (3):

$$G_{l,K}^{\mathrm{alg}} \;=\; \bigcup_{\sigma\in \mathrm{Gal}(L/K)} \rho_l(\tilde{\sigma})\,G_{l,L}^{\mathrm{alg}} \;=\; {\bigsqcup}'_{\sigma\in \mathrm{Gal}(L/K)}\; \rho_l(\tilde{\sigma})\,G_{l,L}^{\mathrm{alg}}. \tag{6.12}$$

where ${\bigsqcup}'_{\sigma\in \mathrm{Gal}(L/K)}$ is the summation over some set of $\sigma \in \mathrm{Gal}(L/K)$ such that $\rho_l(\tilde{\sigma})\,\rho_l(G_L)$ are all different cosets of $\rho_l(G_L)$ in $\rho_l(G_K)$. Because of (6.12) we have $(G_{l,K}^{\mathrm{alg}})^{\circ} \subset G_{l,L}^{\mathrm{alg}}$. It is obvious that $\mathbb{G}_m\,\mathrm{Id}_{V_l} \subset (G_{l,K}^{\mathrm{alg}})^{\circ}$. Hence $\rho_l(\tilde{\gamma})) \in (G_{l,K}^{\mathrm{alg}})^{\circ} \subset G_{l,L}^{\mathrm{alg}}$. Hence $\rho_l(\tilde{\sigma})\,G_{l,L}^{\mathrm{alg}} = i_{L/K}(\rho_l(\tilde{\sigma}\tilde{\gamma}^{-1})\,G_{l,L,1}^{\mathrm{alg}})$ and it follows that $\mathrm{Zar}_{L/K,\,1}$ is an epimorphism. $\square$

Corollary 6.9. *Let L/K be a finite Galois extension. Assume that:*

(1) $L \cap K(\mu_{\overline{l}}^{\otimes\, n}) = K$;
(2) $1 + l\mathbb{Z}_l\,\mathrm{Id}_{V_l} \subset \rho_l(G_K)$;
(3) $\mathrm{Zar}_{L/K}$ *is an isomorphism;*
(4) $G_K/G_L \cong \rho_l(G_K)/\rho_l(G_L)$.

Then each coset of G_K/G_L has the form $\tilde{\sigma}_1\,G_L$ such that:

(1) $\rho_l(\tilde{\sigma}_1) \in \rho_l(G_K)_1$;
(2) $\rho_l(G_K)_1 \;=\; \bigsqcup_{\tilde{\sigma}_1 G_L}\, \rho_l(\tilde{\sigma}_1)\,\rho_l(G_L)_1$;
(3) $G_{l,K,1}^{\mathrm{alg}} \;=\; \bigsqcup_{\tilde{\sigma}_1 G_L}\, \rho_l(\tilde{\sigma}_1)\,G_{l,L,1}^{\mathrm{alg}}$.

Proof. Pick elements $\tilde{\sigma} \in G_K$ which represent all of the cosets of G_L in G_K. Because of assumption (4), we have:

$$\rho_l(G_K) = \bigsqcup_{\tilde{\sigma}\,G_L} \rho_l(\tilde{\sigma})\,\rho_l(G_L). \tag{6.13}$$

By Lemma 6.8, the map $j_{L/K}$ is an isomorphism. Hence for every $\tilde{\sigma}$ there is $\tilde{\sigma}_1 \in G_K$ such that $\rho_l(\tilde{\sigma}_1) \in \rho_l(G_K)_1$ and $\rho_l(\tilde{\sigma})\rho_l(G_L) = \rho_l(\tilde{\sigma}_1)\rho_l(G_L)$. By assumption (4)

we obtain $\tilde{\sigma}\, G_L = \tilde{\sigma}_1\, G_L$. Since $j_{L/K}$ is an isomorphism, the claim (2) holds. The claim (3) follows because $\mathrm{Zar}_{L/K,\,1}$ is an isomorphism by Lemma 6.8. □

THEOREM 6.10. *Assume that:*
(1) $K_e \cap K(\mu_{\bar{l}}^{\otimes\, n}) \,=\, K$*;*
(2) $1 + l\mathbb{Z}_l\, \mathrm{Id}_{V_l} \,\subset\, \rho_l(G_K)$*.*

Then all arrows in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} \rho_l(G_K)/\rho_l(G_{K_e}) & \xrightarrow[\cong]{\mathrm{Zar}_{K_e/K}} & G_{l,K}^{\mathrm{alg}}/G_{l,K_e}^{\mathrm{alg}} \\ {\scriptstyle j_{K_e/K}}\uparrow\cong & & {\scriptstyle i_{K_e/K}}\uparrow\cong \\ \rho_l(G_K)_1/\rho_l(G_{K_e})_1 & \xrightarrow[\cong]{\mathrm{Zar}_{K_e/K,\,1}} & G_{l,K,1}^{\mathrm{alg}}/G_{l,K_e,1}^{\mathrm{alg}} \end{array} \tag{6.14}$$

PROOF. By (5.9) and (5.10), the upper horizontal arrow $\mathrm{Zar}_{K_e/K}$ in diagram (6.14) is an isomorphism. Now the assumptions (1) and (2) and Lemma 6.8 show that all of the arrows in (6.14) are isomorphisms. □

THEOREM 6.11. *Assume that:*
(1) $K_0 \cap K(\mu_{\bar{l}}^{\otimes\, n}) \,=\, K$*,*
(2) $1 + l\mathbb{Z}_l\, \mathrm{Id}_{V_l} \,\subset\, \rho_l(G_K)$*.*

Then all arrows in the following commutative diagram are isomorphisms:

$$\begin{array}{ccc} \rho_l(G_K)/\rho_l(G_{K_0}) & \xrightarrow[\cong]{\mathrm{Zar}_{K_0/K}} & G_{l,K}^{\mathrm{alg}}/G_{l,K_0}^{\mathrm{alg}} \\ {\scriptstyle j_{K_0/K}}\uparrow\cong & & {\scriptstyle i_{K_0/K}}\uparrow\cong \\ \rho_l(G_K)_1/\rho_l(G_{K_0})_1 & \xrightarrow[\cong]{\mathrm{Zar}_{K_0/K,\,1}} & G_{l,K,1}^{\mathrm{alg}}/G_{l,K_0,1}^{\mathrm{alg}} \end{array} \tag{6.15}$$

Moreover each coset of G_K/G_{K_0} *has the form* $\tilde{\sigma}_1\, G_{K_0}$ *such that:*
(1) $\rho_l(\tilde{\sigma}_1) \in \rho_l(G_K)_1$*;*
(2) $\rho_l(G_K)_1 \;=\; \bigsqcup_{\tilde{\sigma}_1 G_{K_0}}\, \rho_l(\tilde{\sigma}_1)\, \rho_l(G_{K_0})_1$*;*
(3) $G_{l,K,1}^{\mathrm{alg}} \;=\; \bigsqcup_{\tilde{\sigma}_1 G_{K_0}}\, \rho_l(\tilde{\sigma}_1)\, G_{l,K_0,1}^{\mathrm{alg}}$*.*

PROOF. It follows from (6.5) and (6.6) that the upper horizontal arrow $\mathrm{Zar}_{K_0/K}$ in diagram (6.15) is an isomorphism. Now the assumptions (1) and (2) and Lemma 6.8 show that all of the arrows in (6.15) are isomorphisms. The isomorphism (6.5) shows that the assumption (4) of Corollary 6.9 is fulfilled, i.e., $G_K/G_{K_0} \cong \rho_l(G_K)/\rho_l(G_{K_0})$. Hence the claims (1)–(3) follow by Corollary 6.9. □

THEOREM 6.12. *Assume Conjecture 5.1* (a) *and assume that for some* l :
(1) $K_0 \cap K(\mu_{\bar{l}}^{\otimes\, n}) \,=\, K$*;*
(2) $1 + l\mathbb{Z}_l\, \mathrm{Id}_{V_l} \,\subset\, \rho_l(G_K)$*;*
(3) $\mathrm{ast}_{l,K}$ *is an isomorphism.*

Then:

$$\mathrm{AST}_{K,\,\mathbb{Q}_l} \;=\; \bigsqcup_{\tilde{\sigma}_1\, G_{K_0}} \rho_l(\tilde{\sigma}_1)\; \mathrm{AST}_{K_0,\,\mathbb{Q}_l} \tag{6.16}$$

$$\mathrm{ST}_K = \bigsqcup_{\tilde{\sigma}_1 \, G_{K_0}} \rho_l(\tilde{\sigma}_1) \, \mathrm{ST}_{K_0} \tag{6.17}$$

In particular the Sato-Tate conjecture (Conjecture 5.9) on the equidistribution of normalized Frobenii in the representation ρ_l with respect to ST_K holds if and only if the conjecture holds for the representation $\rho_l \,|\, G_{K_0}$ with respect to ST_{K_0}.

Proof. By Theorem 6.11 we get

$$G_{l,K,1}^{\mathrm{alg}} = \bigsqcup_{\tilde{\sigma}_1 \, G_{K_0}} \rho_l(\tilde{\sigma}_1) \, G_{l,K_0,1}^{\mathrm{alg}} \,. \tag{6.18}$$

Hence by Proposition 5.7 we get the equality (6.16) which, under base change to $\mathbb{C}$, taking $\mathbb{C}$-points and restricting to maximal compacts, gives the equality (6.17). $\square$

Let us now specialize the previous discussion to abelian varieties.

Remark 6.13. Fix an embedding of K into $\mathbb{C}$. Let (V, ψ) be the Hodge structure associated to an abelian variety A over K (i.e., $n = 1$, $V := H_1(A_{\mathbb{C}}, \mathbb{Q})$, and ψ is the pairing induced by a polarization of A). Take D to be $\mathrm{End}(A_{\overline{F}})_{\mathbb{Q}}$ (noting that this coincides with $D_h = D(V, \psi)$). Let $T_l(A)$ be the l-adic Tate module of A and let $V_l := V_l(A) := T_l(A) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Let ρ_l be the Galois representation of G_K on V_l. In this case, all the assumptions made in chapters 2–5 are satisfied, and the resulting definitions agree with the corresponding definitions made in [**BK**].

J.-P. Serre proved [**Se4**] that the index $e(l)$ of the group of homotheties in $\rho_l(G_K)$ in the group of all homotheties is bounded when l varies. Hence there is $c \in \mathbb{N}$ such that $(\mathbb{Z}_l^{\times})^c \, \mathrm{Id}_{V_l} \subset \rho_l(G_K)$ for all l. Hence for every l coprime to c, we obtain $1 + l\,\mathbb{Z}_l \, \mathrm{Id}_{V_l} \subset \rho_l(G_K)$. In this way, Serre established independence of K_0 from l; an explicit description of K_0 in terms of fields of definition of torsion points was later given by Larsen–Pink [**LP**].

Corollary 6.14. *With notation as in Remark 6.13, suppose that A/F satisfies the Mumford-Tate conjecture, $\mathrm{H}(V, \psi) = \mathrm{L}(V, \psi, D)$, and $\mathrm{DL}_{K_e}(V, \psi, D)$ is connected. Then for $l \gg 0$, the Sato-Tate conjecture holds for A/K with respect to ρ_l, if and only if the conjecture holds for A/K_0 with repect to $\rho_l \,|\, G_{K_0}$.*

Proof. Obviously for $l \gg 0$ the condition (1) of Theorem 6.12 holds. The condition (2) of Theorem 6.12 holds for $l \gg 0$ by the result of Serre [**Se4**] discussed in Remark 6.13 or by the result of Wintenberger [**W**, Corollary 1, p. 5] showing the Lang conjecture. The condition (3) of Theorem 6.12 holds by [**BK**, Theorem 6.1]. $\square$

Corollary 6.15. *With notation as in Remark 6.13, put $g := \dim A$, and let E be the center of D. Assume that either $g \leq 3$ or A is absolutely simple of type I, II or III in the Albert classification with $\frac{g}{de}$ odd, where $d^2 = [D : E]$ and $e := [E : \mathbb{Q}]$. Then for $l \gg 0$, the Sato-Tate conjecture holds for A/K with respect to ρ_l, if and only if the conjecture holds for A/K_0 with repect to $\rho_l \,|\, G_{K_0}$.*

Proof. By [**BGK1**, Theorem 7.12, Cor. 7.19], [**BGK2**, Theorem 5.11, Cor. 5.19] and [**BK**, Theorem 6.11], abelian varieties considered in this corollary satisfy the Mumford-Tate conjecture and the properties: $\mathrm{H}(A) = \mathrm{L}(A)$ and $\mathrm{DL}_{K_e}(A)$ connected. Hence the corollary follows by Corollary 6.14. $\square$

Remark 6.16. Some additional cases for which the conclusion of Corollary 6.15 holds are provided by the Jacobians of (certain) hyperelliptic curves, thanks to the work of Zarhin [**Z1, Z2**].

7. Mumford-Tate group and Mumford-Tate conjecture

For A an abelian variety over K and (V_A, ψ_A) the associated polarized Hodge structure (as in Remark 6.13), there is the following result.

THEOREM 7.1. *(Deligne* [**D1**, I, Prop. 6.2]*, Piatetski-Shapiro* [**P-S**]*, Borovoi* [**Bor**]*; see also* [**Se1**, §4.1]*) For any prime number l,*

$$(G_{l,K}^{\mathrm{alg}})^{\circ} \subseteq \mathrm{MT}(V_A, \psi_A)_{\mathbb{Q}_l}. \tag{7.1}$$

The classical conjecture for A/K states:

CONJECTURE 7.2. *(Mumford-Tate) For any prime number l,*

$$(G_{l,K}^{\mathrm{alg}})^{\circ} = \mathrm{MT}(V_A, \psi_A)_{\mathbb{Q}_l}. \tag{7.2}$$

There is a general Mumford-Tate conjecture in the context of Hodge structures associated with l-adic representations [**UY**].

CONJECTURE 7.3. *(Mumford-Tate) For any prime number l,*

$$(G_{l,K}^{\mathrm{alg}})^{\circ} = \mathrm{MT}(V, \psi)_{\mathbb{Q}_l}. \tag{7.3}$$

Remark 7.4. Assume that analogously to (7.1) there is the following inclusion:

$$(G_{l,K}^{\mathrm{alg}})^{\circ} \subseteq \mathrm{MT}(V, \psi)_{\mathbb{Q}_l}. \tag{7.4}$$

We see that (7.4) is equivalent to the inclusion

$$(G_{l,K,1}^{\mathrm{alg}})^{\circ} \subseteq \mathrm{H}(V, \psi)_{\mathbb{Q}_l}, \tag{7.5}$$

while the Mumford-Tate conjecture is equivalent to the equality

$$(G_{l,K,1}^{\mathrm{alg}})^{\circ} = \mathrm{H}(V, \psi)_{\mathbb{Q}_l}. \tag{7.6}$$

This follows immediately from the following commutative diagram in which every column is exact and every horizontal arrow is a containment of corresponding group schemes. (Recall that n is odd.)

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
(G_{l,K,1}^{\mathrm{alg}})^{\circ} \arrow[r] \arrow[d] & \mathrm{DH}(V,\psi)_{\mathbb{Q}_l} \arrow[r] \arrow[d] & \mathrm{Iso}_{V_l,\psi_l} \arrow[d] \\
(G_{l,K}^{\mathrm{alg}})^{\circ} \arrow[r] \arrow[d] & \mathrm{MT}(V,\psi)_{\mathbb{Q}_l} \arrow[r] \arrow[d] & \mathrm{GIso}_{V_l,\psi_l} \arrow[d] \\
\mathbb{G}_m \arrow[r, "="] \arrow[d] & \mathbb{G}_m \arrow[r, "="] \arrow[d] & \mathbb{G}_m \arrow[d] \\
1 & 1 & 1
\end{tikzcd}$$

The Mumford-Tate and Hodge groups do not behave well in general with respect to products of Hodge structures, as can be seen in the case of abelian varieties [**G**, p. 316]. However, one has the following simple and well-known result (see for instance [**Mo2**, (4.10)]); for more detailed discussions of products, see any of [**Ha, MZ, Z3**].

Theorem 7.5. *The Mumford-Tate groups of Hodge structures have the following properties.*

1. *An isomorphism of rational, polarized Hodge structures* $\alpha : (V_1, \psi_1) \to (V_2, \psi_2)$ *induces isomorphisms* $\mathrm{MT}(V_1, \psi_1) \cong \mathrm{MT}(V_2, \psi_2)$ *and* $\mathrm{H}(V_1, \psi_1) \cong \mathrm{H}(V_2, \psi_2)$.
2. *For* (V, ψ) *is a rational, polarized Hodge structure, let* $(V, \psi)^s := \prod_{i=1}^{s} (V, \psi)$. *Then* $\mathrm{MT}((V, \psi)^s) \cong \mathrm{MT}((V, \psi))$ *and* $\mathrm{H}((V, \psi)^s) \cong \mathrm{H}((V, \psi))$.

One can make a corresponding calculation also on the Galois side.

Theorem 7.6. *We have the following results.*

1. *An isomorphism* $\phi : (V_{1,l}, \psi_{1,l}) \to (V_{2,l}, \psi_{2,l})$ *of* $\mathbb{Q}_l[G_F]$*-modules induces isomorphisms:* $G_{l,K}^{\mathrm{alg}}(V_{1,l}, \psi_{1,l}) \cong G_{l,K}^{\mathrm{alg}}(V_{2,l}, \psi_{2,l})$ *and* $G_{l,K,1}^{\mathrm{alg}}(V_{1,l}, \psi_{1,l}) \cong G_{l,K,1}^{\mathrm{alg}}(V_{2,l}, \psi_{2,l})$.
2. *If* (V_l, ψ_l) *is a* $\mathbb{Q}_l[G_F]$*-module then for any positive integer* s, $G_{l,K}^{\mathrm{alg}}(V_l^s, \psi_l^s) \cong G_{l,K}^{\mathrm{alg}}(V_l, \psi_l)$ *and* $G_{l,K,1}^{\mathrm{alg}}(V_l^s, \psi_l^s) = G_{l,K,1}^{\mathrm{alg}}(V_l, \psi_l)$.

Proof. 1. Obvious.
2. There is a natural isomorphism $\rho_{l,V_l^s} \cong \Delta\rho_{l,V_l}$ in which $\Delta\rho_{l,V_l} : G_K \to \mathrm{GIso}((V_l)^s, \psi_l^s)$ is the natural diagonal representation $\Delta\rho_{l,V_l} = \mathrm{diag}(\rho_{l,V_l}, \ldots, \rho_{l,V_l})$. Hence

$$\rho_{l,V_l^s}(G_K) \cong \Delta\rho_{l,V_l}(G_K) \cong \rho_{l,V_l}(G_K),$$

This gives

$$G_{l,K}^{\mathrm{alg}}(V_l^s, \psi_l^s) \cong \Delta\, G_{l,K}^{\mathrm{alg}}(V_l, \psi_l) \cong G_{l,K}^{\mathrm{alg}}(V_l, \psi_l).$$

Moreover

$$G_{l,K,1}^{\mathrm{alg}}(V_l^s, \psi_l^s) = G_{l,K}^{\mathrm{alg}}(V_l^s, \psi_l^s) \cap \mathrm{Iso}_{((V_l)^s, \psi_l^s)} \cong \Delta\, G_{l,K}^{\mathrm{alg}}(V_l, \psi_l) \cap \mathrm{Iso}_{((V_l)^s, \psi_l^s)}$$

$$\cong G_{l,K}^{\mathrm{alg}}(V_l, \psi_l) \cap \mathrm{Iso}_{((V_l), \psi_l)} = G_{l,K,1}^{\mathrm{alg}}(V_l, \psi_l).$$

□

Corollary 7.7. *If the Mumford-Tate conjecture holds for* V *then it holds for* V^s *for any positive integer* s.

Proof. It follows from Theorems 7.5 and 7.6. □

Remark 7.8. Observe that if the Mumford-Tate conjecture holds for (V, ψ) and K is such that $G_{l,K}^{\mathrm{alg}}$ is connected, then for any $s \geq 1$:

$$G_{l,K,1}^{\mathrm{alg}}((V, \psi)^s) = \mathrm{H}((V, \psi)^s)_{\mathbb{Q}_l}. \tag{7.7}$$

Hence the algebraic Sato-Tate conjecture holds for $(V, \psi)^s$ for any $s \geq 1$ with

$$\mathrm{AST}_K((V, \psi)^s) = \mathrm{H}((V, \psi)^s). \tag{7.8}$$

8. Some conditions for the algebraic Sato-Tate conjecture

Let A be an abelian variety over K and let $D_A := \mathrm{End}_{\overline{F}}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. For the polarized Hodge structure (V_A, ψ_A), the inclusion $\mathrm{H}(V_A, \psi_A) \subseteq \mathrm{L}(V_A, \psi_A, D_A)$ can be strict, which makes the Mumford-Tate conjecture a subtle problem. Mumford [**Mu**] exhibited examples of simple abelian fourfolds for which $\mathrm{H}(V_A, \psi_A) \neq \mathrm{L}(V_A, \psi_A, D_A)$. These examples have trivial endomorphism ring, but the construction was generalized by Pohlmann [**Poh**] to include some abelian varieties of CM type (see [**MZ**] for further discussion). Notwithstanding such constructions, in many cases where A has a large endomorphism algebra as compared to its dimension (e.g., under the hypotheses of Corollary 6.15), one can show that $\mathrm{H}(V_A, \psi_A) = \mathrm{L}(V_A, \psi_A, D_A)$ and that the Mumford-Tate conjecture holds.

Returning to the general case, let $D \subset \mathrm{End}_{\mathbb{Q}}(V)$ be a $\mathbb{Q}$-subalgebra. Let D admit a continuous G_F-action. Let (V, ψ) be a D-equivariant, polarized Hodge structure. Let $(V_l, \psi_l) := (V \otimes_{\mathbb{Q}} \mathbb{Q}_l, \psi \otimes_{\mathbb{Q}} \mathbb{Q}_l)$ be a family of Galois representations associated with the polarized Hodge structure (V, ψ). In this chapter, we assume that the inclusion (7.4) holds. In this setting, we say that the Mumford-Tate conjecture for (V, ψ) is *explained by endomorphisms* if the Mumford-Tate conjecture holds and $\mathrm{H}(V, \psi) = \mathrm{L}(V, \psi, D)$. The following theorem asserts that in cases where the Mumford-Tate conjecture is explained by endomorphisms *and* the twisted decomposable Lefschetz group over $\overline{F}$ is connected, the algebraic Sato-Tate conjecture is in a sense also explained by endomorphisms.

Theorem 8.1. *Assume that the following conditions hold.*

1. $\mathrm{H}(V, \psi) = \mathrm{L}(V, \psi, D) = \mathrm{DL}_{K_e}(V, \psi, D)$.
2. $(G_{l,K}^{\mathrm{alg}})^\circ = \mathrm{MT}(V, \psi)_{\mathbb{Q}_l}$.

Then (5.17) holds for every l. Consequently, the algebraic Sato-Tate conjecture (Conjecture 5.1) holds for (V, ψ) with

$$\mathrm{AST}_K(V, \psi) = \mathrm{DL}_K(V, \psi, D). \tag{8.1}$$

Proof. It is enough to prove (5.19). By our assumptions and Remark 7.4, we get $(G_{l,K_e,1}^{\mathrm{alg}})^\circ = \mathrm{H}(V, \psi)_{\mathbb{Q}_l} = \mathrm{L}(V, \psi, D)_{\mathbb{Q}_l} = \mathrm{DL}_{K_e}(V, \psi, D)_{\mathbb{Q}_l}$. It follows that $\mathrm{DL}_{K_e}(V, \psi, D)_{\mathbb{Q}_l}$ is also connected for every l, and by (5.14) we obtain $(G_{l,K_e,1}^{\mathrm{alg}})^\circ = G_{l,K_e,1}^{\mathrm{alg}}$ for every l. □

Remark 8.2. Under the assumptions of Theorem 8.1, the results of Theorems 7.5, 7.6 and 3.5 show that the algebraic Sato-Tate conjecture holds for V^s for all $s \geq 1$ with $\mathrm{AST}_K(V^s, \psi^s) = \mathrm{DL}_K(V^s, \psi^s, M_s(D)) \cong \mathrm{DL}_K(V, \psi, D) = \mathrm{AST}_K(V, \psi, D)$.

Conversely, if the algebraic Sato-Tate conjecture for (V, ψ, D) is explained by endomorphisms, so is the Mumford-Tate conjecture.

Theorem 8.3. *Assume that (5.17) and (8.1) hold for every l (so in particular, the algebraic Sato-Tate conjecture holds). Moreover, assume that $(G_{l,K}^{\mathrm{alg}})^\circ \subset \mathrm{MT}(V, \psi)_{\mathbb{Q}_l}$. We then have the following.*

1. $\mathrm{H}(V, \psi) = \mathrm{L}(V, \psi, D)$.
2. $(G_{l,K}^{\mathrm{alg}})^\circ = \mathrm{MT}(V, \psi)_{\mathbb{Q}_l}$.

Proof. By our assumptions and Remark 7.4 (see (7.5)), we have

$$(G_{l,K,1}^{\mathrm{alg}})^\circ \subseteq \mathrm{H}(V, \psi)_{\mathbb{Q}_l} \subseteq \mathrm{L}(V, \psi, D)_{\mathbb{Q}_l} = \mathrm{DL}_K(V, \psi, D)^\circ_{\mathbb{Q}_l}. \tag{8.2}$$

By (8.1) we get

$$(G_{l,K,1}^{\mathrm{alg}})^{\circ} = \mathrm{H}(V,\psi)_{\mathbb{Q}_l} = \mathrm{L}(V,\psi,D)_{\mathbb{Q}_l} = \mathrm{DL}_K(V,\psi,D)^{\circ}_{\mathbb{Q}_l}. \tag{8.3}$$

Hence by Remark 7.4, we obtain $(G_{l,K}^{\mathrm{alg}})^{\circ} = \mathrm{MT}(V,\psi)_{\mathbb{Q}_l}$. Moreover, since $\mathrm{H}(V,\psi)$ is closed in $\mathrm{L}(V,\psi,D)$, (8.3) gives

$$\mathrm{H}(V,\psi) = \mathrm{L}(V,\psi,D). \tag{8.4}$$

□

Remark 8.4. Recall that

$$\mathrm{L}(V,\psi,D) = \mathrm{DL}_K(V,\psi,D)^{\circ} \ \triangleleft\ \mathrm{DL}_K^{\mathrm{id}}(V,\psi,D) \ \triangleleft\ \mathrm{DL}_K(V,\psi,D). \tag{8.5}$$

Consider the following epimorphism of groups:

$$\mathrm{DL}_K(V,\psi,D)/\,\mathrm{L}(V,\psi,D) \ \to\ \mathrm{DL}_K(V,\psi,D)/\,\mathrm{DL}_K^{\mathrm{id}}(V,\psi,D) \ \cong\ G(K_e/K). \tag{8.6}$$

If (V,ψ,D) satisfies the assumptions of Theorem 8.1, then the epimorphism (8.6) is an isomorphism. In this case we have an identification

$$\pi_0(\mathrm{AST}_K(V,\psi,D)) \ \cong\ \mathrm{Gal}(K_e/K). \tag{8.7}$$

9. Motivic Galois group and motivic Serre group

In the following sections we will give construction of the general algebraic Sato-Tate group in the category of motives for absolute Hodge cycles. See [**DM**] (cf. [**Ja1**], [**Pan**], [**Se2**]) concerning the construction and properties of the category of motives for absolute Hodge cycles. We will also make the ℓ-adic realization of this construction explicit, and show that if a suitably motivic form of the Mumford-Tate conjecture holds then the algebraic Sato-Tate conjecture holds as well.

Remark 9.1. The category of motives for absolute Hodge cycles enjoys very nice properties: it is a semisimple abelian category and its Hom's are finite-dimensional $\mathbb{Q}$-vector spaces. It is mainly due to the fact that the definition of Hom's is explained via the Betti, étale and de Rham realizations [**DM**, Prop. 6.1, p. 197]. The advantage of use of this category of motives is that we do not need to assume standard conjectures in our constructions.

Definition 9.2. Let K be a number field. Choose an embedding of K into $\overline{K}$. Let $\mathcal{M}_K$ (resp. $\mathcal{M}_{\overline{K}}$) (see [**DM**]) be the motivic category for absolute Hodge cycles over K (resp. $\overline{K}$). The Betti realization defines the fiber functor H_B :

$$H_B : \mathcal{M}_K \to \mathrm{Vec}_{\mathbb{Q}}. \tag{9.1}$$

The functor H_B factors through the functor

$$\mathcal{M}_K \to \mathcal{M}_{\overline{K}}, \qquad M \mapsto \overline{M} := M \otimes_K \overline{K}. \tag{9.2}$$

For $M \in \mathcal{M}_K$ let $\mathcal{M}_K(M)$ denote the smallest Tannakian subcategory of $\mathcal{M}_K$ containing M. Let $H_B|\mathcal{M}_K(M)$ be the restriction of H_B to $\mathcal{M}_K(M)$.

Definition 9.3. The motivic Galois groups are defined as follows [**DM**], [**Se2**]:

$$G_{\mathcal{M}_K} := \mathrm{Aut}^{\otimes}(H_B), \tag{9.3}$$

$$G_{\mathcal{M}_K(M)} := \mathrm{Aut}^{\otimes}(H_B|\mathcal{M}_K(M)). \tag{9.4}$$

The algebraic groups $G_{\mathcal{M}_K(M)}$ are reductive but not necessarily connected (see [**DM**, Prop. 2.23, p. 141], cf. [**DM**, Prop. 6.23, p. 214], [**Se2**, p. 379]). Observe that the finite-dimensional $\mathbb{Q}$-vector space $\mathrm{Hom}_{\mathcal{M}_{\overline{K}}}(\overline{M}, \overline{N}) \in \mathcal{M}_K^0$ is a discrete G_K-module, so we consider it as an Artin motive. Recall that $\mathcal{M}_K^0$ is equivalent to $\mathrm{Rep}_{\mathbb{Q}}(G_K)$, the category of finite-dimensional $\mathbb{Q}$-vector spaces with continuous actions of G_K.

Definition 9.4. Fix a motive M and put:

$$D := D(M) := \mathrm{End}_{\mathcal{M}_{\overline{K}}}(\overline{M}) \tag{9.5}$$

Let $h^0(D)$ denote the Artin motive corresponding to D. Let $\mathcal{M}_K^0(D)$ be the smallest Tannakian subcategory of $\mathcal{M}_K^0$ containing $h^0(D)$ and put:

$$G_{\mathcal{M}_K^0(D)} := \mathrm{Aut}^{\otimes}(H_B^0|\mathcal{M}_K^0(D)). \tag{9.6}$$

There is a natural embedding of motives [**DM**, p. 215], [**Ja1**, p. 53]:

$$h^0(D) \subset \underline{\mathrm{End}}_{\mathcal{M}_K(M)}(M) = \underline{\mathrm{End}}_{\mathcal{M}_K}(M). \tag{9.7}$$

Recall that $\underline{\mathrm{End}}_{\mathcal{M}_K(M)}(M) = M^\vee \otimes M \in \mathcal{M}_K(M)$. In addition $G_{\mathcal{M}_K^0} \cong G_K$, so we observe that

$$G_{\mathcal{M}_K^0(D)} \cong \mathrm{Gal}(K_e/K).$$

Since $\mathcal{M}_K$ is semisimple [**DM**, Prop. 6.5] and $\mathcal{M}_K(M)$ is a strictly full subcategory of $\mathcal{M}_K$, the motive $h^0(D)$ splits off of $\underline{\mathrm{End}}_{\mathcal{M}_K(M)}(M)$ in $\mathcal{M}_K$. Moreover the semisimplicity of $\mathcal{M}_K$, together with the observation that $\mathcal{M}_K^0$ and $\mathcal{M}_K(M)$ are strictly full subcategories of $\mathcal{M}_K$, shows that the top horizontal and left vertical maps in the following diagram are faithfully flat (see [**DM**, (2.29)]):

$$\begin{tikzcd} G_{\mathcal{M}_K} \arrow[r, two heads] \arrow[d] & G_K \arrow[d] \\ G_{\mathcal{M}_K(M)} \arrow[r, two heads] & \mathrm{Gal}(K_e/K) \end{tikzcd} \tag{9.8}$$

In particular all homomorphisms in (9.8) are surjective.

In the construction of $\mathcal{M}_K$ [**DM**, p.200–203] one starts with **effective motives** $h(X)$ and morphisms between them:

$$\mathrm{Hom}_{\mathcal{M}_K}(h(X), h(Y)) := \mathrm{Mor}^0_{\mathrm{AH}}(X, Y) := \mathrm{CH}^d_{\mathrm{AH}}(X \times Y) \tag{9.9}$$

where X and Y are smooth projective over K and X is of pure dimension d. This leads swiftly (via Karoubian envelope construction etc.) to the definition of the motivic category for absolute Hodge cycles $\mathcal{M}_K$. In particular $\mathrm{Hom}_{\mathcal{M}_K}(M, N)$, for any $M, N \in \mathcal{M}_K$, are relatively easy to handle. The obvious grading of the cohomology ring brings the decomposition of the identity on $h(X)$ into a sum of the natural projectors:

$$\mathrm{id}_{h(X)} = \sum_{i \geq 0} \pi^i \tag{9.10}$$

As a result we get the natural decomposition [**DM**, p. 201–202]:

$$h(X) = \bigoplus_{i \geq 0} h^i(X) \tag{9.11}$$

where $h^i(X) := (h(X), \pi^i)$. See also [**Ja1**] and [**Pan**] for additional information about $\mathcal{M}_K$.

Since $\mathcal{M}_K$ is abelian and semisimple, every motive $M \in \mathcal{M}_K$ is a direct summand of $h(X)(m)$, the twist of $h(X)$ by the m-th power of the Lefschetz motive $\mathbb{L} := h^2(\mathbb{P}^1)$ for some $m \in \mathbb{Z}$. The direct summands of motives of the form $h^r(X)(m)$ will be called *homogeneous motives*. Let L/K be a field extension such that $K \subset L \subset \overline{K}$. Then in $\mathcal{M}_L$, the motive $M \otimes_K L \in \mathcal{M}_L$ is a direct summand of the motive $h(X \otimes_K L)(m)$.

Observe that $H_B|\mathcal{M}_K(h^r(X))\ (h^r(X)) = H_B(h^r(X)) = H^r(X(\mathbb{C}), \mathbb{Q})$ and $V := H^r(X(\mathbb{C}), \mathbb{Q})$ admits a $\mathbb{Q}$-rational polarized Hodge structure of weight r with polarization ψ^r. The polarization comes up as follows. It is shown in [**DM**, pp. 197–199] (cf. [**Pan**, p. 478–480], [**Ja1**, pp. 2–4]) that if dim $X = d$, then there is an element $\psi^r \in \mathrm{CH}^{2d-r}_{\mathrm{AH}}(X \times X)$ such that for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, ψ^r induces a $\mathbb{Q}$-bilinear map:

$$\psi^r\ :\ H^r_\sigma(X(\mathbb{C}), \mathbb{Q}) \times H^r_\sigma(X(\mathbb{C}), \mathbb{Q}) \to \mathbb{Q}(-r) \tag{9.12}$$

which gives the polarization $\psi^r_{\mathbb{R}} := \psi^r \otimes_{\mathbb{Q}} \mathbb{R}$ of the real Hodge structure:

$$\psi^r_{\mathbb{R}}\ :\ H^r_\sigma(X(\mathbb{C}), \mathbb{R}) \times H^r_\sigma(X(\mathbb{C}), \mathbb{R}) \to \mathbb{R}(-r). \tag{9.13}$$

It is then shown [**DM**, Prop. 6.1 (e), p. 197] that the Hodge decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$ is $D = D(M)$-equivariant for $M = h^r(X)$.

In effect, for any homogeneous motive $M \in \mathcal{M}_K$, this induces the polarization of the real Hodge structure associated with the rational Hodge structure on the Betti realization $V := H_B(M)$. The Hodge decomposition of $V \otimes_{\mathbb{Q}} \mathbb{C}$ is again $D = D(M)$-equivariant.

From now on in this paper, M will always denote a homogeneous motive.

By the definition and properties of $\mathrm{Aut}^{\otimes}(H_B|\mathcal{M}_K(M))$, cf. [**DM**, p. 128–130] and computations in [**DM**, p. 198–199], we have:

$$G_{\mathcal{M}_K(M)} \subset \mathrm{GIso}_{(V,\psi)}\,. \tag{9.14}$$

Definition 9.5. Define the following algebraic groups:

$$G_{\mathcal{M}_K(M),1} := G_{\mathcal{M}_K(M)} \cap \mathrm{Iso}_{(V,\psi)}$$
$$G^{\circ}_{\mathcal{M}_K(M),1} := (G_{\mathcal{M}_K(M)})^{\circ} \cap \mathrm{Iso}_{(V,\psi)}\,.$$

The algebraic group $G_{\mathcal{M}_K(M),1}$ will be called the *motivic Serre group*.

Remark 9.6. Serre denotes the group $G_{\mathcal{M}_K(M),1}$ by $G^1_{\mathcal{M}_K(M)}$ [**Se2**, p. 396].

Definition 9.7. For any $\tau \in \mathrm{Gal}(K_e/K)$, put

$$\mathrm{GIso}^{\tau}_{(V,\psi)} := \{g \in \mathrm{GIso}_{(V,\psi)} :\ g\beta g^{-1} = \rho_e(\tau)(\beta)\ \ \forall \beta \in D\}. \tag{9.15}$$

We have:

$$\bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} \mathrm{GIso}^{\tau}_{(V,\psi)} \subset \mathrm{GIso}_{(V,\psi)}\,. \tag{9.16}$$

Observe that

$$\mathrm{GIso}^{\mathrm{id}}_{(V,\psi)} = C_D(\mathrm{GIso}_{(V,\psi)}). \tag{9.17}$$

Remark 9.8. The bottom horizontal arrow in the diagram (9.8) is

$$G_{\mathcal{M}_K(M)} \to G_{\mathcal{M}_K(D)} \cong \mathrm{Gal}(K_e/K). \tag{9.18}$$

Let $g \in G_{\mathcal{M}_K(M)}$ and let $\tau := \tau(g)$ be the image of g via the map (9.18). Hence for any element $\beta \in D$ considered as an endomorphism of V we have:

$$g\beta g^{-1} = \rho_e(\tau)(\beta). \tag{9.19}$$

Definition 9.9. For any $\tau \in \mathrm{Gal}(K_e/K)$, put

$$G^{\tau}_{\mathcal{M}_K(M)} := \{g \in G_{\mathcal{M}_K(M)} :\ g\beta g^{-1} = \rho_e(\tau)(\beta),\ \ \forall \beta \in D\}. \tag{9.20}$$

It follows from (9.19), (9.20), and the surjectivity of (9.18) that

$$G_{\mathcal{M}_K(M)} \ = \bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} G^{\tau}_{\mathcal{M}_K(M)} \tag{9.21}$$

It is clear from (9.14) and (9.15) that

$$G^{\tau}_{\mathcal{M}_K(M)} \subset \mathrm{GIso}^{\tau}_{(V,\psi)}\,. \tag{9.22}$$

Hence (9.19) and (9.21) give

$$(G_{\mathcal{M}(M)})^{\circ} \triangleleft G^{\mathrm{id}}_{\mathcal{M}_K(M)} \triangleleft G_{\mathcal{M}_K(M)}. \tag{9.23}$$

The map (9.18) gives the following natural map:

$$G_{\mathcal{M}_K(M),1} \to \mathrm{Gal}(K_e/K). \tag{9.24}$$

Definition 9.10. For any $\tau \in \mathrm{Gal}(K_e/K)$ put

$$G^{\tau}_{\mathcal{M}_K(M),1} := \{g \in G_{\mathcal{M}_K(M),1} :\ g\beta g^{-1} = \rho_e(\tau)(\beta),\ \ \forall \beta \in D\}. \tag{9.25}$$

It follows that there is the following equality

$$G^{\tau}_{\mathcal{M}_K(M),1} = G_{\mathcal{M}_K(M),1} \cap G^{\tau}_{\mathcal{M}_K(M)}. \tag{9.26}$$

Let $\tau \in \mathrm{Gal}(K_e/K)$. By (3.2), (3.3), (9.22) we have

$$G^{\tau}_{\mathcal{M}_K(M),1} \subset DL^{\tau}_K(V,\,\psi,D) \tag{9.27}$$

$$G_{\mathcal{M}_K(M),1} \subset DL_K(V,\,\psi,D). \tag{9.28}$$

The equality (9.21) gives:

$$G_{\mathcal{M}_K(M),1} = \bigsqcup_{\tau \in \mathrm{Gal}(K_e/K)} G^{\tau}_{\mathcal{M}_K(M),1} \tag{9.29}$$

Hence:

$$(G_{\mathcal{M}_K(M),1})^{\circ}\ \triangleleft\ G^{\mathrm{id}}_{\mathcal{M}_K(M),1}\ \triangleleft\ G_{\mathcal{M}_K(M),1}, \tag{9.30}$$

so (9.26) gives:

$$G_{\mathcal{M}_K(M),1}/\,G^{\mathrm{id}}_{\mathcal{M}_K(M),1}\ \subset\ G_{\mathcal{M}_K(M)}/\,G^{\mathrm{id}}_{\mathcal{M}_K(M)}. \tag{9.31}$$

Remark 9.11. The l-adic representation

$$\rho_l \, : \, G_K \to \mathrm{GL}(V_l) \tag{9.32}$$

associated with M factors through $G_{\mathcal{M}_K(M)}(\mathbb{Q}_l)$ (see [**Pan**, Corollary p. 473–474] cf. [**Se2**, p. 386]). Hence

$$G_{l,K}^{\mathrm{alg}} \subset G_{\mathcal{M}_K(M)_{\mathbb{Q}_l}} \tag{9.33}$$

where $G_{\mathcal{M}_K(M)_{\mathbb{Q}_l}} := G_{\mathcal{M}_K(M)} \otimes_{\mathbb{Q}} \mathbb{Q}_l$.

10. Motivic Mumford-Tate and Motivic Serre groups

Since X/K is smooth projective and hence proper, Remarks 4.1 and 4.2 show that $V_l := H^r(\overline{X}, \mathbb{Q}_l)$, the l-adic realization of the motive $h^r(X)$, is of Hodge-Tate type. Hence the image of the representation ρ_l, contains an open subset of homotheties of the group $\mathrm{GL}(V_l)$ [**Su**, Prop. 2.8], and similarly for any Tate twist such that $H^r(\overline{X}, \mathbb{Q}_l(m))$ has nonzero weights.

Remark 10.1. In the previous statement, the assumption of nonzero weights is essential. Indeed, if X has dimension d, then $H^{2d}(\overline{X}, \mathbb{Q}_l(d)) \cong \mathbb{Q}_l$ as G_K-modules. Hence the action of G_K on $H^{2d}(\overline{X}, \mathbb{Q}_l(d))$ is trivial, so the image of the Galois representation is a trivial group and hence does not contain homotheties.

From now until the end of the paper, let $M \in \mathcal{M}_K$ be a motive which is a direct summand of a motive of the form $h^r(X)(m)$. We assume that the l-adic realization of $h^r(X)(m)$ has nonzero weights with repect to the G_K-action. The l-adic realization of $\overline{M}$ is a $\mathbb{Q}_l[G_F]$-direct summand of the l-adic realization of $h^r(X)(m)$. Hence the l-adic representation corresponding to $V_l := H_l(\overline{M})$ has image that contains an open subgroup of homotheties.

In the following commutative diagram, all horizontal arrows are closed immersions and the columns are exact.

$$\begin{tikzcd}
1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] \\
G_{l,K,1}^{\mathrm{alg}} \arrow[r] \arrow[d] & G_{\mathcal{M}_K(M),1_{\mathbb{Q}_l}} \arrow[r] \arrow[d] & \mathrm{Iso}_{(V_l,\psi_l)} \arrow[d] \\
G_{l,K}^{\mathrm{alg}} \arrow[r] \arrow[d] & G_{\mathcal{M}_K(M)_{\mathbb{Q}_l}} \arrow[r] \arrow[d] & \mathrm{GIso}_{(V_l,\psi_l)} \arrow[d] \\
\mathbb{G}_m \arrow[r, "="] \arrow[d] & \mathbb{G}_m \arrow[r, "="] \arrow[d] & \mathbb{G}_m \arrow[d] \\
1 & 1 & 1
\end{tikzcd}$$

In particular it follows that:

$$G_{l,K,1}^{\mathrm{alg}} \subset (G_{\mathcal{M}_K(M),1})_{\mathbb{Q}_l}. \tag{10.1}$$

We have the following analogue of Theorem 4.8.

Theorem 10.2. *Assume that $G^{\circ}_{\mathcal{M}_K(M),1}$ is connected. Then the following map is an isomorphism:*

$$i_M \ : \ \pi_0(G_{\mathcal{M}_K(M),1}) \xrightarrow{\cong} \pi_0(G_{\mathcal{M}_K(M)}).$$

Proof. We will write $\mathcal{M}(M)$ for $\mathcal{M}_K(M)$ in the following commutative diagram to make notation simpler.

$$\begin{tikzcd}
 & 1 \arrow[d] & 1 \arrow[d] & 1 \arrow[d] & \\
1 \arrow[r] & (G_{\mathcal{M}(M),1})^{\circ} \arrow[r] \arrow[d] & G_{\mathcal{M}(M),1} \arrow[r] \arrow[d] & \pi_0(G_{\mathcal{M}(M),1}) \arrow[r] \arrow[d, "i_M"] & 1 \\
1 \arrow[r] & (G_{\mathcal{M}(M)})^{\circ} \arrow[r] \arrow[d] & G_{\mathcal{M}(M)} \arrow[r] \arrow[d] & \pi_0(G_{\mathcal{M}(M)}) \arrow[r] \arrow[d] & 1 \\
1 \arrow[r] & \mathbb{G}_m \arrow[r, "="] \arrow[d] & \mathbb{G}_m \arrow[r] \arrow[d] & 1 & \\
 & 1 & 1 & &
\end{tikzcd}$$

By definition the rows are exact. The middle column is exact by the definition of $G_{\mathcal{M}_K(M),1}$ and the exactness of the middle column in the previous diagram. Hence the map i_M is surjective. Since $G^{\circ}_{\mathcal{M}_K(M),1}$ has the same dimension as $G_{\mathcal{M}_K(M),1}$ and by assumption $G^{\circ}_{\mathcal{M}_K(M),1}$ is connected, we then have $G^{\circ}_{\mathcal{M}_K(M),1} = (G_{\mathcal{M}_K(M),1})^{\circ}$. Hence the left column is also exact. This shows that i_M is an isomorphism. □

Remark 10.3. Since $G_{\mathcal{M}_K(M)}$ is reductive, the middle vertical column of the diagram of the proof of Theorem 10.2 shows that $G_{\mathcal{M}_K(M),1}$ is also reductive.

Corollary 10.4. *Assume that $G^{\circ}_{\mathcal{M}_K(M),1}$ is connected. Then there are natural isomorphisms*

(10.2)
$$G_{\mathcal{M}_K(M),1}/\, G^{\mathrm{id}}_{\mathcal{M}_K(M),1} \xrightarrow{\cong} G_{\mathcal{M}_K(M)}/\, G^{\mathrm{id}}_{\mathcal{M}_K(M)},$$
(10.3)
$$G^{\mathrm{id}}_{\mathcal{M}_K(M),1}/\, (G_{\mathcal{M}_K(M),1})^{\circ} \xrightarrow{\cong} G^{\mathrm{id}}_{\mathcal{M}_K(M)}/\, (G_{\mathcal{M}_K(M)})^{\circ},$$
(10.4)
$$G_{\mathcal{M}_K(M),1}/\, G^{\mathrm{id}}_{\mathcal{M}_K(M),1} \xrightarrow{\cong} \mathrm{DL}_K(V,\psi,D)/\ \mathrm{DL}^{\mathrm{id}}_K(V,\psi,D) \xrightarrow{\cong} \mathrm{Gal}(K_e/K).$$

In particular the natural map (9.24) is surjective.

Proof. This follows from (8.6), (9.23), (9.30), (9.31), the surjectivity of (9.18) and Theorem 10.2. □

Definition 10.5. The algebraic groups:

$$\mathrm{MMT}_K(M) := G_{\mathcal{M}_K(M)}$$
$$\mathrm{MS}_K(M) := G_{\mathcal{M}_K(M),1}$$

will be called the *motivic Mumford-Tate group* and (as before) the *motivic Serre group* for M respectively.

Conjecture 10.6. *(Motivic Mumford-Tate) For any prime number l,*

$$G_{l,K}^{\mathrm{alg}} = \mathrm{MMT}_K(M)_{\mathbb{Q}_l}. \tag{10.5}$$

By the diagram above Theorem 10.2, Conjecture 10.6 is equivalent to the following.

Conjecture 10.7. *(Motivic Sato-Tate) For any prime number l,*

$$G_{l,K,1}^{\mathrm{alg}} = \mathrm{MS}_K(M)_{\mathbb{Q}_l}. \tag{10.6}$$

Remark 10.8. Conjecture 10.6 is equivalent to the conjunction of the following equalities:

$$(G_{l,K}^{\mathrm{alg}})^\circ = (\mathrm{MMT}_K(M)_{\mathbb{Q}_l})^\circ \tag{10.7}$$

$$\pi_0(G_{l,K}^{\mathrm{alg}}) = \pi_0(\mathrm{MMT}_K(M)_{\mathbb{Q}_l}). \tag{10.8}$$

Similarly, Conjecture 10.7 is equivalent to the conjunction of the following equalities:

$$(G_{l,K,1}^{\mathrm{alg}})^\circ = (\mathrm{MS}_K(M)_{\mathbb{Q}_l})^\circ \tag{10.9}$$

$$\pi_0(G_{l,K,1}^{\mathrm{alg}}) = \pi_0(\mathrm{MS}_K(M)_{\mathbb{Q}_l}). \tag{10.10}$$

11. The algebraic Sato-Tate group

As in the previous section, we work with motives M which are direct summands of motives of the form $h^r(X)(m)$; in this section, we propose a candidate for the algebraic Sato-Tate group for such motives. We prove, under the assumption in Definition 11.7, that our candidate for algebraic Sato-Tate group is the expected one. In particular the assumption of Definition 11.7 holds if M is an AHC motive (see Definition 11.3 and Remark 11.4).

Remark 11.1. One observes ([**Pan**, Corollary p. 473–474], cf. [**Se2**, p. 379]) that

$$\mathrm{MT}(V, \psi) \subset (G_{\mathcal{M}_K(M)})^\circ \tag{11.1}$$

Hence we get:

$$\mathrm{H}(V, \psi) \subset (G_{\mathcal{M}_K(M),1})^\circ \tag{11.2}$$

Recall that $C_D(\mathrm{Iso}_{(V,\psi)}) = \mathrm{DL}_K^{\mathrm{id}}(V, \psi, D)$. It follows by (9.17), (9.22), (9.23), and (11.1) we get:

$$\mathrm{MT}(V, \psi) \subset (G_{\mathcal{M}_K(M)})^\circ \subset G_{\mathcal{M}_K(M)}^{\mathrm{id}} \subset C_D(\mathrm{GIso}_{(V,\psi)}). \tag{11.3}$$

Similarly by (9.27), (9.30), and (11.2) that:

$$\mathrm{H}(V, \psi) \subset (G_{\mathcal{M}_K(M),1})^\circ \subset G_{\mathcal{M}_K(M),1}^{\mathrm{id}} \subset C_D(\mathrm{Iso}_{(V,\psi)}). \tag{11.4}$$

Remark 11.2. Observe that (11.3) gives an approximation for $\pi_0(G_{\mathcal{M}_K(M)}^{\mathrm{id}})$ and (11.4) gives an approximation for $\pi_0(G_{\mathcal{M}_K(M),1}^{\mathrm{id}})$.

We observe that for n odd the equality

$$\mathrm{H}(V,\psi) = C_D(\mathrm{Iso}_{(V,\psi)}) \tag{11.5}$$

is equivalent to the following equality:

$$\mathrm{MT}(V,\psi) = C_D(\mathrm{GIso}_{(V,\psi)}). \tag{11.6}$$

DEFINITION 11.3. A motive $M \in \mathcal{M}_K$ will be called an *AHC motive* if every Hodge cycle on any object of $\mathcal{M}_K(M)$ is an absolute Hodge cycle (cf. [**D1**, p. 29], [**Pan**, p. 473]).

Remark 11.4. J-P. Serre conjectured [**Se2**, sec. 3.4] the equality $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$. By [**DM**] the conjecture holds for abelian varieties A/K and for AHC motives M (cf. [**Pan**, Corollary p. 474]).

Remark 11.5. In [**Se2**, p. 380] there are examples of the computation of $\mathrm{MMT}_K(M) = G_{\mathcal{M}_K(M)}$. In [**BK**, Theorems 7.3, 7.4], we compute $\mathrm{MMT}_K(M)$ for abelian varieties of dimension ≤ 3 and families of abelian varieties of type I, II and III in the Albert classification.

If Serre's conjecture $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$ holds for M, then by (9.33) the containment (7.4) holds:

$$(G_{l,K}^{\mathrm{alg}})^\circ \subset \mathrm{MT}(V,\psi)_{\mathbb{Q}_l} \tag{11.7}$$

and for n odd it is equivalent to:

$$(G_{l,K,1}^{\mathrm{alg}})^\circ \subset \mathrm{H}(V,\psi)_{\mathbb{Q}_l}. \tag{11.8}$$

In particular (11.7) and (11.8) hold for AHC motives (cf. Remark 11.4).

Remark 11.6. To obtain $G_{l,K}^{\mathrm{alg}}$ as an extension of scalars to $\mathbb{Q}_l$ of an expected algebraic Sato-Tate group defined over $\mathbb{Q}$, the assumption in the following definition is natural in view of (11.2), (11.8), Theorem 10.2 and Remark 11.4.

DEFINITION 11.7. Assume that $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$. Then the *algebraic Sato-Tate group* $\mathrm{AST}_K(M)$ is defined as follows:

$$\mathrm{AST}_K(M) := \mathrm{MS}_K(M). \tag{11.9}$$

Every maximal compact subgroup of $\mathrm{AST}_K(M)(\mathbb{C})$ will be called a *Sato-Tate group* associated with M and denoted $\mathrm{ST}_K(M)$.

THEOREM 11.8. *Assume that we have* $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$. *Then the group* $\mathrm{AST}_K(M)$ *is reductive and:*

$$\mathrm{AST}_K(M) \subset \mathrm{DL}_K(V,\psi,D), \tag{11.10}$$

$$\mathrm{AST}_K(M)^\circ = \mathrm{H}(V,\psi) \tag{11.11}$$

$$\pi_0(\mathrm{AST}_K(M)) = \pi_0(\mathrm{MMT}_K(M)), \tag{11.12}$$

$$\pi_0(\mathrm{AST}_K(M)) = \pi_0(\mathrm{ST}_K(M)). \tag{11.13}$$

$$G_{l,K,1}^{\mathrm{alg}} \subset \mathrm{AST}_K(M)_{\mathbb{Q}_l}, \ \textit{i.e. Conjecture } 5.1\,(\mathrm{a}) \textit{ holds for } M. \tag{11.14}$$

Proof. The group $\mathrm{AST}_K(M)$ is reductive by Remark 10.3. Moreover (11.10) is just (9.28). By assumption $\mathrm{MT}(V,\psi) = (G_{\mathcal{M}_K(M)})^\circ$ and the equality $\mathrm{DH}(V,\psi) = \mathrm{H}(V,\psi)$ (which holds n odd), we have:

$$G^\circ_{\mathcal{M}_K(M),1} = (G_{\mathcal{M}_K(M)})^\circ \cap \mathrm{Iso}_{(V,\psi)} = \mathrm{MT}(V,\psi) \cap \mathrm{Iso}_{(V,\psi)} = \mathrm{H}(V,\psi). \tag{11.15}$$

Hence $G^\circ_{\mathcal{M}_K(M),1}$ is connected and

$$\mathrm{AST}_K(M)^\circ = (G_{\mathcal{M}_K(M),1})^\circ = G^\circ_{\mathcal{M}_K(M),1}, \tag{11.16}$$

so (11.11) follows. The equality (11.12) follows directly from the Theorem 10.2. Equality (11.13) follows since $\mathrm{AST}_K(M)^\circ(\mathbb{C})$ is a connected complex Lie group and any maximal compact subgroup of a connected complex Lie group is a connected real Lie group. (11.14) follows by (10.1) and the assumption (see also Definitions 10.5 and 11.7). □

Corollary 11.9. *Under the assumptions that* $\mathrm{MT}(V,\psi) = \mathrm{MMT}_K(M)^\circ$ *and* $\mathrm{DH}(V,\psi) = \mathrm{H}(V,\psi)$*, there are the following commutative diagrams with exact rows:*

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{H}(V,\psi) \arrow[r] \arrow[d] & \mathrm{AST}_K(M) \arrow[r] \arrow[d] & \pi_0(\mathrm{AST}_K(M)) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{L}(V,\psi,D) \arrow[r] & \mathrm{DL}_K(V,\psi,D) \arrow[r] & \pi_0(\mathrm{DL}_K(V,\psi,D)) \arrow[r] & 0 \end{tikzcd} \tag{11.17}$$

$$\begin{tikzcd} 0 \arrow[r] & \pi_0(G^{\mathrm{id}}_{\mathcal{M}_K(M),1}) \arrow[r] \arrow[d] & \pi_0(\mathrm{AST}_K(M)) \arrow[r] \arrow[d] & \mathrm{Gal}(K_e/K) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & \pi_0(\mathrm{DL}^{\mathrm{id}}_K(V,\psi,D)) \arrow[r] & \pi_0(\mathrm{DL}_K(V,\psi,D)) \arrow[r] & \mathrm{Gal}(K_e/K) \arrow[r] & 0 \end{tikzcd} \tag{11.18}$$

Proof. The exactness of the top row of the Diagram (11.17) follows from (11.11). The exactness of the top row of the Diagram (11.18) follows immediately from Corollary 10.4. □

Corollary 11.10. *Assume that* $\mathrm{H}(V,\psi) = C_D(\mathrm{Iso}_{(V,\psi)})$*. Then*

$$\mathrm{AST}_K(M) = \mathrm{DL}_K(V,\psi,D). \tag{11.19}$$

Proof. It follows by the assumption and (11.4) that

$$\pi_0(G^{\mathrm{id}}_{\mathcal{M}_K(M),1}) = \pi_0(\mathrm{DL}^{\mathrm{id}}_K(V,\psi,D)) = 1.$$

Hence the middle vertical arrow in the diagram (11.18), which is the right vertical arrow in the diagram (11.17), is an isomorphism. Since $\mathrm{L}(V,\psi,D) = (C_D\,\mathrm{Iso}_{(V,\psi)})^\circ$, by assumption we have $\mathrm{H}(V,\psi) = \mathrm{L}(V,\psi,D)$. Hence the left vertical arrow in the diagram (11.17) is an isomorphism, and so the middle vertical arrow in the diagram (11.17) is an isomorphism. □

Corollary 11.11. *If* $\mathrm{H}(V,\psi) = C_D(\mathrm{Iso}_{(V,\psi)})$ *and the Mumford-Tate conjecture holds for* M*, then the algebraic Sato-Tate conjecture holds:*

$$G^{\mathrm{alg}}_{l,K,1} = \mathrm{AST}_K(M)_{\mathbb{Q}_l}.$$

Proof. By (10.1) and Corollary 11.10:

$$G_{l,K,1}^{\mathrm{alg}} \subset \mathrm{AST}_K(M)_{\mathbb{Q}_l} = \mathrm{DL}_K(V,\psi,D)_{\mathbb{Q}_l}.$$

By the assumption $\mathrm{H}(V,\psi) = \mathrm{DL}_{K_e}(V,\psi,D)$. By virtue of the equivalence of (5.19) and (5.17), we only need to prove that $(G_{l,K,1}^{\mathrm{alg}})^{\circ} = \mathrm{H}(V,\psi)_{\mathbb{Q}_l}$ which is equivalent to the Mumford-Tate conjecture by Remark 7.4 . $\square$

Remark 11.12. Theorem 11.8 and its Corollaries 11.9, 11.10 and 11.11 show that $\mathrm{AST}_K(M)$ from Definition 11.7 is a natural candidate for the algebraic Sato-Tate group for the motive M.

Remark 11.13. Let M be a homogeneous motive which is a direct summand of $h^i(X)(m)$. Put $W := H^i(X(\mathbb{C}), \mathbb{Q}(m))$. If ψ is the polarization of the Hodge stucture on W then we will also denote by ψ the induced polarization on $V = H_B(M)$ (see Chapter 9). Observe also that $W_l := H_{et}^i(\overline{X}, \mathbb{Q}_l(m))$. We will denote by ρ_{W_l} the natural representation $\rho_{W_l} : G_K \to \mathrm{GIso}_{(W_l,\psi_l)}(\mathbb{Q}_l)$.

Theorem 11.14. *Let M be a motive that is a summand of $h^i(X)(m)$ with nonzero weights. Let the Hodge structure associated with M have pure odd weight n. Assume that Conjecture 5.1* (a) *holds for M and there is $c \in \mathbb{N}$ such that $(\mathbb{Z}_l^{\times})^c \, \mathrm{Id}_{W_l} \subset \rho_{W_l}(G_K)$ for all l. Moreover assume that for some l coprime to c :*

(1) $K_0 \cap K(\mu_{\bar{l}}^{\otimes n}) = K$,
(2) $\mathrm{ast}_{l,K}$ *is an isomorphism with respect to ρ_l.*

Then the Sato-Tate Conjecture holds for the representation $\rho_l : G_K \to \mathrm{GIso}_{(V_l,\psi_l)}(\mathbb{Q}_l)$ with respect to $\mathrm{ST}_K(M)$ if and only if it holds for $\rho_l : G_{K_0} \to \mathrm{GIso}_{(V_l,\psi_l)}(\mathbb{Q}_l)$ with respect to $\mathrm{ST}_{K_0}(M)$.

Proof. Because V_l is a subquotient of W_l as a $\mathbb{Q}_l[G_K]$-module, we have $(\mathbb{Z}_l^{\times})^c \, \mathrm{Id}_{V_l} \subset \rho_l(G_K)$ for all l. Since l is coprime to c then $1 + l\,\mathbb{Z}_l \subset (\mathbb{Z}_l^{\times})^c$. Hence the assumptions in this theorem guarantee that all assumptions of Theorem 6.12 are satisfied. Hence Theorem 11.14 follows by Theorem 6.12. $\square$

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